FINITE GROUPS WITHOUT ELEMENTS OF ORDER $2p$ FOR AN ODD PRIME p

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Vavilov Memorial 2024

September 18, 2024, Saint Petersburg, Russia

I shall speak about some problems and results on the study of finite groups without elements of order $2p$ for an odd prime p.

Some definitions and notation

Let G be a finite group. Denote by $\pi(G)$ the set of all prime divisors of the order of G and by $\omega(G)$ the set of orders of the elements of G. The Gruenberg–Kegel graph, or the prime graph, of G is the graph $\Gamma(G)$ with vertex set $\pi(G)$ in which two vertices p and q are adjacent if and only if either $p \neq q$ and $pq \in \omega(G)$. Let $s = s(G)$ be the number of connected components of $\Gamma(G)$ and let $\{\pi_i(G) \mid 1 \leq i \leq s(G)\}$ be the set of connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we always suppose that $2 \in \pi_1(G)$. A set of a pairwise adjacent vertices of a graph is called its clique. A set of a pairwise non-adjacent vertices of a graph is called its coclique. Denote by $t(G)$ the greatest cardinality of cocliques of $\Gamma(G)$ and by $t(2, G)$ the greatest cardinality of cocliques containing 2 of $\Gamma(G)$.

Recall that a finite group X is almost simple whenever $S \leq X \leq Aut(S)$ for some finite nonabelian simple group S; equivalently, provided that the socle of X (i. e., the product of all minimal non-trivial normal subgroups of X) is a finite nonabelian simple group. A finite group X is quasisimple whenever $X' = X$ and $X/Z(X)$ is a finite nonabelian simple group.

Given a positive integer n and a prime p denote by np the p-part of n which is the greatest power of p dividing n. A semidirect product of groups A and B is denoted by $A \rtimes B$, while $A{\cdot}B$ denotes a nonsplit extension of A by B.

Given a finite group G and some set π of primes, denote by $O_\pi(G)$ the largest normal π-subgroup of G, and by $O^\pi(G)$ the smallest normal subgroup of G the quotient of G over which is a π-group. For brevity, put $O(G) = O_{2'}(G)$.

We will also use the notation $L_n^\epsilon(q)$, $PGL_n^\epsilon(q)$ и $S_{2n}(q)$, where $\epsilon \in \{+,-\}$ and $L_n^+(q) = L_n(q) = PSL_n(q)$, $L_n^-(q) = U_n(q) = PSU_n(q)$, $PGL_n^+(q) = PGL_n(q)$, $PGU_n^-(q) = PGU_n(q)$, and $S_{2n}(q) = PSp_{2n}(q)$.

Put $S = L_2(q)$, where $q = p^{2k}$ for some odd prime p and $k \in \mathbb{N}$. Put $PGL_2^*(q) = S\langle\delta\varphi\rangle$, where $PGL_2(q) = S\langle\delta\rangle$ and φ is the field automorphism of order 2 of S. It is well known that $PGL_2^*(q) \setminus S$ contains no involutions.

Motivation

It is very-known the dominating role of involutions in the thery of finite non-solvable groups and especially in the classification of finite simple groups (CFSG). If G is a finite group of even order and $2 \neq p \in \pi(G)$ then it is important to know whether the vertices 2 and p are adjacent in the graph $\Gamma(G)$. We shall call an W_p-group for an odd prime p a finite group of order dividing by $2p$ and having no elements of order $2p$; the vertices 2 and p in the Gruenberg–Kegel graph of such group are non-adjacent. Denote by W the class of all W_p-groups when p runs all odd primes. Then W coincides with the class of finite groups G such that $t(2, G) \geq 2$.

By Theorem 7.1 from the paper of Vasil'ev and Vdovin [Algebra i Logika, 2005], any finite simple group belongs to the class W, except the alternating group of degree n such that the numbers n, $n-1$, $n-2$ and $n-3$ are non-prime. It is clear that finite groups with disconnected Gruenberg–Kegel graphs belong to W. Thus, the class W is wide and interesting and the following problem arises naturally.

Problem 1. *Describe W_p-groups for an odd prime p, at least small ones.*

Given a finite group G and an odd prime p, denote by $W_p(G)$ the group $O^{\{2,p\}'}(G/O_{\{2,p\}'}(G))$ It is clear that G is a W_p-group if and only if $W_p(G)$ is a W_p-group.

The history of results on particular cases of Problem 1 is very rich.

$EPPO$-groups

Burnside [Trans. Cambridge Phil. Soc., 1900] considered the case when the order of any element of a finite group either is odd or equals 2.

G. Higman [J. London Math. Soc. (2), 1957] described finite groups whose element orders are prime powers (Shi called such groups shortly as $EPPO$-groups). It is clear that the connected components of the Gruenberg-Kegel graph of a $EPPO$-group are one-element.

C_{pp}-groups

A finite group of order dividing by a prime p is called a C_{pp}-group if the centralizer of any its non-trivial element is a p-group. It is clear that a $EPPO$-group G is C_{pp}-group for any $p \in \pi(G)$.

Back at the dawn of classification, M. Suzuki in his pioneer fundamental papers [Trans. Amer. Math. Soc., 1961], [Ann. Math., 1962] obtained a description of C_{22}-groups, i. e. groups which are W_p-groups for for all odd prime divisors p of their orders. In the sequal, this description was refined and reduced to a criterion in works of G. Higman [Odd characterizations of finite simple groups: lecture notes, 1968], P. Martino [Amer. J. Math. Soc., 1972], Stewart [Proc. London Math. Soc. (3), 1973] and Brandl [Bol. Un. Mat. Ital. (5), 1981]. As a corollary, a complete description of $EPPO$-groups was obtained.

A complete description of non-primary C_{33}-groups is obtain by G. Higman [Odd characterizations of finite simple groups: lecture notes, 1968], Stewart [Proc. London Math. Soc. (3), 1973] and Fletcher, Stellmacher, and Stewart [Quart. J. Math. Oxford. Ser. (2), 1977].

A description of non-primary C_{55}-groups is obtained in the papers by Dolphi, Jabara, Lucido [Siberian Mat. Zh., 2004] and Astill, C. Parker and Waldecker [Sib. Mat. Zh., 2012]. Note that the simple C_{55}-groups was determined previously in the papers by Williams [J. Algebra, 1981], AK [Mat. Sb., 1989] and Chen and Shi [J. Southwest Normal Univ., 1993].

It is well-known the following particular case of Problem 1.

Problem 2. *Describe C_{pp}-groups for a prime $p > 5$.*

Groups with disconnected prime graph

It is clear that Gruenberg–Kegel graph of any non-primary C_{pp}-group is disconnected.

The first result about finite groups with disconnected prime graph is the following structural theorem obtained by Gruenberg and Kegel about 1975 in an unpublished paper. The proof of this theorem was published in the paper of Williams [J. Algebra, 1981], post-graduate of Gruenberg.

Gruenberg-Kegel Theorem. *If G is a finite group with disconnected Gruenberg-Kegel graph, then one of the following holds:*

(a) *G is a Frobenius group;*

(b) *G is a 2-Frobenius group, i. e., $G = ABC$, where A and AB are normal subgroups of G, AB and BC are Frobenius groups with kernels A and B and complements B and C, respectively;*

(c) *G is an extension of a nilpotent $\pi_1(G)$-group by a group A, where $Inn(P) \leq A \leq Aut(P)$, P is a finite simple group with $s(G) \leq s(P)$, and A/P is a $\pi_1(G)$-group.*

According to Suzuki, a proper subgroup H of a group G is called *isolated subgroup (or CC-subgroup)* in G if $C_G(h) \leq H$ for all non-trivial element $h) \in H$. It is easily to understand that an isolated subgroup H in a finite group G is $\pi(H)$-Hall subgroup in G. It is well-known that core and complement in a Frobenius group are its isolated subgroups. Finite groups having an isolated subgroup are studied without the classification of finite simple groups by many known algebraists (Burnside, Frobenius, Suzuki, Feit, Thompson, G. Higman, Arad, Chillag, Busarkin, Gorchakov, Podufalov and others).

Williams [J. Algebra, 1981] established a relation between connected components of the graph $\Gamma(G)$ and isolated subgroups of odd order in finite non-solvable group G.

Williams Theorem. *If G is a finite non-solvable group with known composition factors, then, for any $i > 1$, the group G contains a nilpotent isolated $\pi_i(G)$-Hall subgroup $X_i(G)$.*

Gruenberg—Kegel and Williams Theorems imply that the class of finite groups with disconnected Gruenberg-Kegel graph coincides with the class of finite groups having an isolated subgroup.

Gruenberg-Kegel Theorem implies the complete description of solvable finite group with disconnected Gruenberg-Kegel graph (they are groups from items (a) or (b) of the theorem) and shows also that the question of the studying a non-solvable finite group with disconnected Gruenberg-Kegel graph is reduced largely to the studying some properties of simple non-abelian groups.

The following conjecture was posed.

Gruenberg-Kegel conjecture. *There exists an upper bound for the numbers $s(G)$, where G is a finite group.*

Gruenberg-Kegel Theorem implies that maximum for the numbers $s(G)$ is achieved on simple groups G. Williams [J. Algebra, 1981] obtained an explicit description of connected components of the Gruenberg-Kegel graph for all finite simple non-abelian groups except the groups of Lie type of even characteristic.

AK [Mat. Sbornik, 1989] obtained such description for the remaining case of the groups of Lie type of even characteristic. Now this work is my most cited paper.

Later my result was repeated by Iiyori and Yamaki [J. Algebra, 1993] in connection with an application of Gruenberg-Kegel graph to the proof of well-known Frobenius conjecture.

But later, some inaccuracies in all three papers were found. So, famous M. Suzuki indicated on a mistake in the paper of Iiyori and Yamaki. They in [J. Algebra, 1996] corrected the mistake, but some their other inaccuracies are remained.

In a joint work of AK and Mazurov [Siberian Math. J., 2000], the corresponding tables were corrected and described the subgroups $X_i(G)$ for $i > 1$ from Williams Theorem.

As a consequence, Gruenberg-Kegel conjecture is true: $s(G) \leq 6$ for any finite group G.

The classification of connected components of Gruenberg-Kegel graph for finite simple groups were applied by Lucido [Rend. Sem. Mat. Univ. Padova, 1999, 2002] for obtaining analogous classification for all finite almost simple groups.

The following natural problem arises.

Problem 3. *Study the finite non-solvable groups with disconnected Gruenberg-Kegel graph, which are not almost simple.*

Problem 3 is solved for several particular cases only, because here some non-trivial problems related with modular representations of finite almost simple groups arise. Problem 3 largely reduces to the following complicated problem, which is of independent interest.

Problem 4. *For a finite simple group G and a given prime r, describe all irreducible $GF(r)G$-modules V such that an element x of prime order p ($p \neq r$) from G acts on V fixed-point-freely, i. e., $C_V(x) = \{0\}$.*

Vasil'ev Theorem

In 2005, A. V. Vasil'ev observed that the proof of the Gruenberg-Kegel Theorem uses essentially the fact that a finite group G with disconnected Gruenberg–Kegel graph contains an element of odd prime order which does not adjanced in Γ to 2. It turns out that the condition on $\Gamma(G)$ to be disconnected could in many cases be successfully replaced by the weaker condition of non-adjacency of 2 and some odd prime divisor of order of G. In [Sib. Mat. Zh., 2005], A. V. Vasil'ev proved a wide generalization of the Gruenberg-Kegel Theorem for non-solvable groups. This result was a few sharpened by A. V. Vasil'ev in [Sib. Mat. Zh., 2009].

Vasil'ev Theorem. *Let G be a finite non-solvable group such that $t(2, G) \geq 2$. Then G is an extension of a solvable group K by an almost simple group A with the socle S for which $t(S) \leq t(G) - 1$, and one of the following statements holds:*

(1) *there exists a prime $r \in \pi(K)$ non-adjacent in $\Gamma(G)$ to 2, $S \cong A_7$ or $L_2(q)$ for some odd q, $t(G) = t(2, S) = 3$ and $t(2, G) = 2$;*

(2) *for every prime $r \in \pi(G)$ non-adjacent in $\Gamma(G)$ to 2, a Sylow r-subgroup of G is isomorphic to a Sylow r-subgroup of S; in particular, $t(2, S) \geq t(2, G)$.*

Vasil'ev Theorem can be considered as a general structural theorem for non-solvable groups from class W with an emphasis on a relation with their Gruenberg-–Kegel graphs.

Our results

Now we consider our results on Problem 1.

The non-abelian finite simple W_3-groups were determined in 1977 in the three independent articles of Podufalov [Algebra i Logika], Fletcher, Stellmacher, and Stewart [Quart. J. Math. Oxford. Ser. (2)], as well as Gordon [Bull. Austral. Math. Soc.]. The problem of describing general finite W_3-groups remained open for more than 40 years before AK and Minigulov [Mat. Zametki, 2018] solved it without using the classification of finite simple groups. Their results have already been applied to study finite groups with certain properties of the prime graph.

AK-Minigulov Theorem. *Let G be a W_3-group Then one of the following assertions holds:*

(1) *$G/O(G)$ is a cyclic or (generalized) quaternion 2-group, a Sylow 3-subgroup in $O(G)$ is abelian, and $O(G)$ has 3-length 1;*

(2) *$G/O_{3'}(G)$ is a cyclic 3-group or the dihedral group of order $2|G|_3$, the degree of nilpotence of a Sylow 2-subgroup in $O_{3'}(G)$ does not exceed 2, and $O_{3'}(G)$ is a solvable group of 2-length not exceeding 1;*

(3) *the group $O^{2,3'}(G/O_{2,3'}(G))$ is isomorphic to one of the groups $L_2(2^n)$, $L_2(3^n)$, $PGL_2(3^n)$, $L_2(3^n).2_3$ (n is even), $L_2(q)$ ($q \equiv \pm 5 (mod 12)$), $L_3(2^n)$ ($(2^n - 1)_3 \leq 3$), $U_3(2^n)$($(2^n + 1)_3 \leq 3$), and the extension of a non-trivial elementary abelian 2-group E by the group $L_2(2^n)$) for E isomorphic, as an $GF(2^n)L_2(2^n)$-module, to a direct sum of natural $GF(2^n)L_2(2^n)$-modules.*

For solvable W_p-groups and $p > 3$, we solve Problem 1 by the following theorem (see [Sib. Mat. Zh. 65:4, 2024]).

Theorem 1. *Let G be a solvable W_p-group for $p > 3$. Then one of the following assertions holds:*

(1) *the group $G/O(G)$ is isomorphic to cyclic or (generalized) quaternion 2-group, $SL_2(3)$ or $SL_2(3){\cdot}2$, a Sylow p-subgroup in $O(G)$ is abelian, and $O(G)$ has p-length 1;*

(2) *$G/O_{p'}(G)$ is a cyclic p-group or a Frobenius group with cyclic core of order $|G|_p$ and a cyclic complement of order dividing $p-1$, the degree of nilpotence of a Sylow 2-subgroup in $O_{p'}(G)$ does not exceed $(p^2-1)/4$ (and this estimate is exact for $p = 5$) and $O_{p'}(G)$ has p-length not exceeding 1.*

Using Theorem 1, we strengthen Vasil'ev Theorem with an emphasis on normal structure of investigated groups proving the following theorem.

Theorem 2. *Let G be a non-solvable W_p-group for an odd prime p, $K = S(G)$ and $\overline{G} = G/K$. Then $\overline{G}$ is an almost simple group with the socle S, p does not divide the index $|\overline{G} : S|$, $t(S) \geq t(G) - 1$, $t(2, S) \geq t(2, G)$, and one of the following assertions holds:*

(1) *$K = O_{\{2,p\}'}(G))$;*

(2) *p divides $|K|$, a Sylow 2-subgroup of G is (generalized) quaternion group, $G/O(G)$ is isomorphic to either $2{\cdot}A_7$ or an extension of $SL_2(q)$ for odd $q > 3$ by a cyclic group of either odd order or a doubled odd order, $O_{p',p,p'}(O(G)) = O(G)$, an involution from $K = Z^*(G)$ inverts some (abelian) Sylow p-subgroup of $O(G)$ and centralizes $O(G)/O_{p',p}(O(G))$, $t(G) = t(2, S) = 3$, and $t(2, G) = 2$;*

(3) *p does not divide $|K|$, 2 divides $|K|$, a Sylow p-subgroup of G is cyclic, $O_{2',2,2'}(K) = K$ and S centralizes $K/O_{2',2}(K)$.*

Remark. In the case (3) of Theorem 2, the last term E of the derived series of $G/O_{2',2}(K)$ is a quasisimple W_p-group such that $EK/O_{2',2}(K)$ is a central product of $K/O_{2',2}(K)$ and E, $E/Z(E)$ is isonorphic to S, and E acts faithfully on $O_{2',2}(K)/O(K)$.

Therefore, Theorems 1 and 2 reduce largely a solving Problem 1 to investigating the cases (1) and (3) of Theorem 2, i. e., almost simple W_p-groups and faithful 2-modular representations of quasisimple W_p-groups. In connection with this fact, some Problems, which are of independent interest, arise.

Problem 4. *Determine all almost simple W_p-groups for $p \geq 5$.*

He and Shi in [Int. J. Group Theory, 2022] determined all simple W_5-groups.

Using this result of He and Shi, we determined recently in [Sib. Mat. Zh. 65:4, 2024] all almost simple W_5-groups. We proved the following theorem.

Theorem 3. *Let G be finite almost simple group with the socle L and 5 divides $|G|$. Then G is a W_5-group if and only if one of the following holds:*

(1) *the group $O^{\{2,5\}'}(G)$ is isomorphic to $L_2(5^f)$ with $f > 1$, $PGL_2(5^f)$ with $f > 1$, or to $PGL_2^*(5^f)$ with even f;*

(2) *$L \cong L_2(q)$, where $q \equiv \epsilon 1 \pmod 5$ for $\epsilon \in \{+, -\}$ with $(q - \epsilon 1)_2 \leq 2$, and either $O^{\{2,5\}'}(G) = L$ or $O^{\{2,5\}'}(G) \cong PGL_2^*(q)$, where q is the square of an odd integer, or $O^{\{2,5\}'}(G) = L \rtimes \langle t \rangle$, where q is a square, $\epsilon = -$, and t the field automorphism of order2 of L;*

(3) *$L \cong L_3^\epsilon(q)$, where $\epsilon \in \{+, -\}$ and $q \equiv -\epsilon 1 \pmod 5$ with even q, and either $O^{\{2,5\}'}(G) = L$ or q is a square, $\epsilon = +$, and $L \rtimes \langle d^{-1}t^d \rangle \leq O^{\{2,5\}'}(G) \leq PGL_3(q) \rtimes \langle d^{-1}td \rangle$, or $O^{\{2,5\}'}(G) = L \rtimes \langle t \rangle$, where t is respectively the field or graph-field automorphism of order2 of L, while d is some element of $Inndiag(L)$;*

(4) *$L \cong L_4^\epsilon(q)$, where $\epsilon \in \{+, -\}$ and $q \equiv \pm 2 \pmod 5$ with $(q + \epsilon 1)_2 \leq 2$, as well as $O^{\{2,5\}'}(G) = L$;*

(5) *$L \cong L_5^\epsilon(q)$, where $\epsilon \in \{+, -\}$ and $q \equiv \pm 2 \pmod 5$ with even q, as well as $O^{\{2,5\}'}(G) = L$;*

(6) *$L \cong S_4(q)$, where $2 < q \equiv \pm 2 \pmod 5$, and $O^{\{2,5\}'}(G) = L$;*

(7) *$L \cong Sz(q)$, where $q > 2$, and $O^{\{2,5\}'}(G) = L$;*

(8) *$G = L \cong A_7,\ M_{11},\ M_{22}$ или M_{23}.*

We have also the following important problem.

Problem 5. *For a finite quasisimple W_p-group G, describe all irreducible $GF(2)G$-modules V such that an element of order p from G acts on V fixed-point-freely.*

Note that the case of Problem 5, when G is a group of Lie type over a finite field of characteristic 2, is most difficult. The study of this case is closely connected with the study of 2-modular irreducible representations of simple algebraic groups over an algebraically closed field of characteristic 2.

Extending and refining Problem 1 we obtain the following

Problem 6. *Let G be a finite W_p-group for an odd prime p, $Q = O_2(G) \neq 1$, $\overline{G} = G/Q$ a finite quasisimple W_p-group. The following questions arise.*

1) *What are the chief factors of the group G involving to Q as $\overline{G}$-modules?*

2) *What is the structure of the group Q (isomorphic type, nilpotency class, exponent, derived length etc.)?*

3) *If Q is elementary abelian group, is the action of $\overline{G}$ on Q completely irreducible?*

4) *Is the extension of G over Q splittable?*

The results we have obtained can be applied to obtaining new arithmetical characterizations of finite groups, in particular, to studying finite groups by properties of their Gruenberg-Kegel graphs.