SOME PROBLEMS AROUND $\mathrm{SL}_2(\mathbb{C})$

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Words and groups

Let F_n be the free group of rank n and let G be a group.
The word map
$$\widetilde{w} : G^n \to G$$
is defined by the formula $\widetilde{w}(g_1, \dots, g_n) := w(g_1, \dots, g_n)$.

Examples

1. $n = 1, w = x^m,\ G = F_n$. Question: $\mid F_n/\langle \mathrm{Im}\, \widetilde{w}\rangle \mid < \infty$? (the Burnside problem).
2. $n = 2, w = [x, y], G$ is a finite simple group. Question: $\mathrm{Im}\ \widetilde{w} = G$? (the Ore problem).

Word maps on $\mathrm{SL}_2(\mathbb{C})$

A. Borel (1983): Let G be a semisimple algebraic group and let $w \neq e$. Then $\overline{\mathrm{Im}\ \widetilde{w}} = G$.
Example: $G = \mathrm{SL}_2(\mathbb{C}),\ w = x^2,\ \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \notin \mathrm{Im}\ \widetilde{w}$.

T.Bandman, Yu. Zarhin (2016): *Let* $G = \mathrm{SL}_2(\mathbb{C}),\ e \neq w \in F_n$. *Then every semisimple element of* G *is contained in* $\mathrm{Im}\ \widetilde{w}$.

In the same paper: *If* $w \notin F_n^2 = [[F_n, F_n], [F_n, F_n]]$, *then* $\mathrm{Im}\ w = G$.

F. Gnutov, N.Gordeev (2021):
There exists a sequence $\{w_n\}_{n=1}^{\infty}$ *such that* $w_n \in F_2^n,\ \mathrm{Im}\ \widetilde{w}_n = \mathrm{PGL}_n(\mathbb{C})$.
U. Jezernik, J Sanchez-Hernandez (2022):
If
$$w = [[x^m, y^n], [x^k, y^l]] \text{ for any } m, n, k, l,$$
then $\mathrm{Im}\ \widetilde{w} = \mathrm{PGL}_n(\mathbb{C})$.

The variety of representations

Let $\widetilde{w} : G^n \to G$ be a word map and let $e \in G$ be the neutral element of G and let $\mathcal{W}_w = \widetilde{w}^{-1}(e)$. Then
$$w(g_1, \dots, g_n) = e \text{ for every sequence } (g_1, \dots, g_n) \in \mathcal{W}_w$$
and the group $\langle g_1, \dots, g_n\rangle$ is a factorgroup of the group $\Gamma_w := F_n/\langle\langle w\rangle\rangle$. If $\rho : \Gamma_w \to G$ is a homomorphism of the group $\Gamma_w = \langle \bar{x}_1, \dots, \bar{x}_n\rangle = F_n/\langle\langle w\rangle\rangle$, where $\bar{x}_i$ are the images of the generators x_i of F_n in the factorgroup Γ_w, then $(\rho(\bar{x}_1), \dots, \rho(\bar{x}_n)) \in \mathcal{W}_w$.

Thus there is a bijection

$$\mathcal{W}_w \leftrightarrow \mathrm{Hom}\big(\Gamma_w, G\big).$$

Let $X \subset F_n$ and let $\mathcal{W}_X = \cap_{w\in X}\mathcal{W}_w \subset G^n$ then there is a bijection

$$\mathcal{W}_X \leftrightarrow \mathrm{Hom}\big(F_n/\langle\langle X\rangle\rangle, G\big).$$

In the case $G = \mathrm{SL}_n(\mathbb{C})$ the set $\mathcal{W}_X$ is a closed subset of G^n and it is called *the variety of representations* of the group $F_n/\langle\langle X\rangle\rangle$.
In general, $\mathcal{W}_X$ is reducible, $\mathcal{W}_X = \bigcup_{i=1}^m \mathcal{W}_X^i$ where $\mathcal{W}_X^i$ are irreducible components of W_X.
The sets $\{\mathcal{W}_X^i\}$ are invariant under conjugation by the elements of G and therefore there exist the algebraic factors $\{\mathcal{W}_X^i//G\}_{i=1}^m$ and $W_X//G = \bigcup_{i=1}^m \mathcal{W}_X^i//G$.

Platonov's Problem. Describe the groups $\Gamma = F_n/\langle X\rangle$ that satisfy the condition $\dim \mathcal{W}_X//G = 0$?
Platonov's Conjecture (1986). $\dim \mathcal{W}_X//G = 0 \Rightarrow \Gamma$ is an arithmetical subgroup of a simple algebraic group.
Counterexample by H.Bass, A. Lubotzky (2000)

Let $G = \mathrm{SL}_2(\mathbb{C}),\ X = \{w\},\ e \neq w \in F_2 = \langle x, y\rangle,\ \mathcal{W}_w \subset G^2$.

$$\dim \mathcal{W}_w^i \leq 5 \text{ for every irreducible component } \mathcal{W}_w^i.$$

N.Gordeev, B.Kunyavski, E. Plotkin (2016):
If $\dim \mathcal{W}_w^i < 5$ *for some* i*, then a unipotent element* $e \neq u \in \mathrm{Im}\,\widetilde{w}$ *and therefore* $G \setminus Z(G) \subset \mathrm{Im}\,\widetilde{w}$*.*

Question: $G \setminus Z(G) \subset \mathrm{Im}\,\widetilde{w} \overset{??}{\Rightarrow} \dim \mathcal{W}_w^i < 5$ for some i.

The representation of one-relator subgroups in $\mathrm{SL}_2(\mathbb{C})$

Let $w \in F_2, \Gamma_w = F_2/\langle\langle w\rangle\rangle,\ G = \mathrm{SL}_2(\mathbb{C})$,

$$\mathrm{Hom}(\Gamma_w, G) = \mathcal{W}_w = \bigcup_{i=1}^m \mathcal{W}_w^i.$$

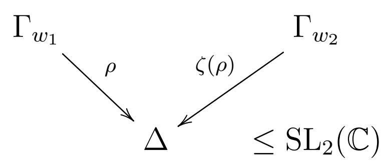
Definition. *A subset* Y *of a topological space* X *is called almost open in* Y *if* $Y = Y \setminus \big(\bigcup_{j=1}^\infty Y_j\big)$ *where* $\{Y_j\}$ *is a countable set of closed subsets of* X*.*

Proposition. *For every* $i = 1, \ldots, m$ *there exists a non-empty almost open subset* $\mathcal{W}_w^{*i}$ *such that for every pair of points* $(g_1, g_2), (g_1', g_2') \in W_i^{*i}$ *the groups* $\langle g_1, g_2\rangle, \langle g_1', g_2'\rangle$ *are isomorphic to the same factorgroup* Γ_w^i *of* Γ_w*.*

G-equivalence of words

$$\mathbf{w_1} \asymp_{\mathbf{G}} \mathbf{w_2} \Leftrightarrow \mathcal{W}_{\mathbf{w_1}} = \mathcal{W}_{\mathbf{w_2}}.$$

$$w_1 \asymp_G w_2 \Leftrightarrow \mathrm{Hom}(\Gamma_{w_1}, G) \overset{\zeta}{\leftrightarrow} \mathrm{Hom}(\Gamma_{w_2}, G)$$



Example. $w_1 = [x, [x^2, yxy^{-1}] \asymp_G w_2 = [x^2, yxy^{-1}]$.

G-order of words

$$\mathbf{w_1} \succcurlyeq_{\mathbf{G}} \mathbf{w_2} \Leftrightarrow \mathcal{W}_{\mathbf{w_1}} \supseteqq \mathcal{W}_{\mathbf{w_2}}.$$

Proposition. *Let $w, \omega \in F_2$ then:*
i. $\omega \in \langle\langle w \rangle\rangle \Rightarrow \omega \succcurlyeq_G w$*; in particular,*

$$w^a \succcurlyeq_G w, \;\; [w, w'] \succcurlyeq_G w$$

for every $a \in \mathbb{Z}, \; w' \in F_2$*;*

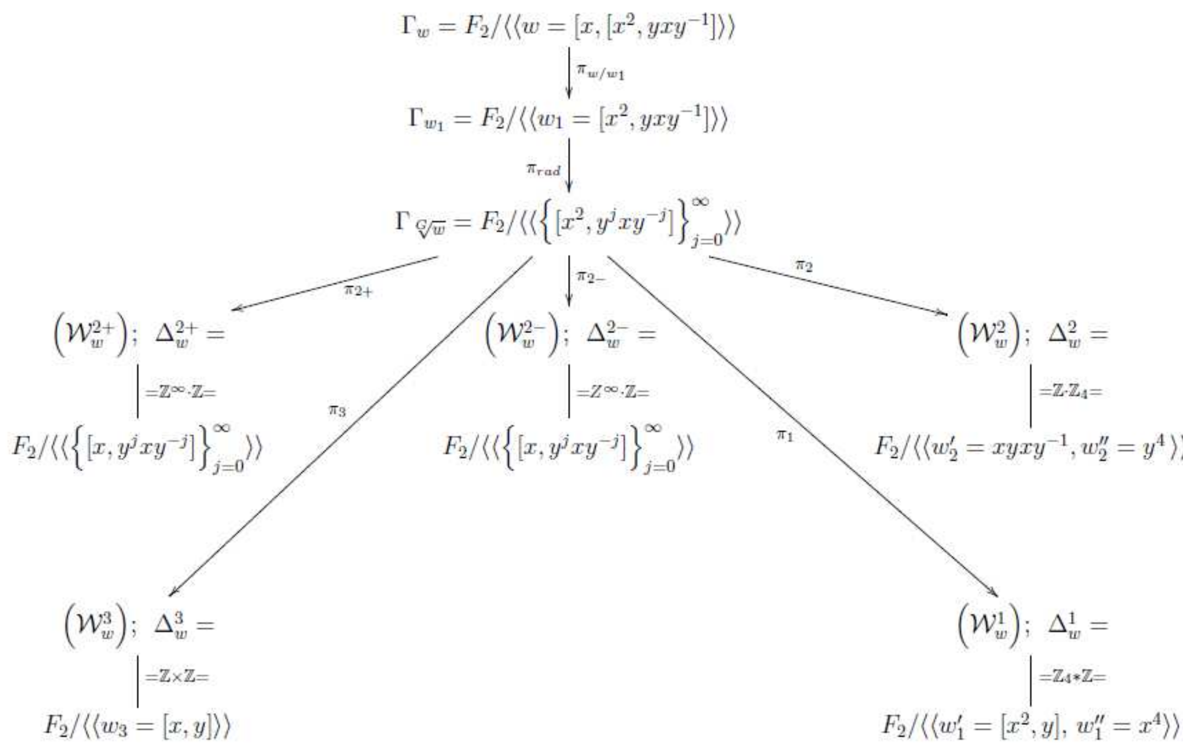
ii. if $\rho_x : \Gamma_w \to G$ *is a faithful representation for some* $x \in \mathcal{W}_w$*, then* $\langle\langle w \rangle\rangle = \{\omega \in F_2 \;\; | \;\; \omega \succcurlyeq_G w\}$*;*

The radical of a word

$\sqrt[G]{w} := \langle \{\omega \in F_2 \;\; | \;\; \omega \succcurlyeq_G w\} \rangle$;
$\sqrt[G]{w}^{\,i} := \langle \{\omega \in F_2 \; | \; \mathcal{W}_\omega \supset \mathcal{W}_w^i\} \rangle$.

Theorem.
i. The set $\{(g_1, g_2) \in \mathcal{W}_w^i \;\; | \;\; \langle g_1, g_2 \rangle \approx F_2/\sqrt[G]{w}^{\,i}\}$ *is an almost open subset* $\neq \emptyset$ *of* $\mathcal{W}_w^i$ *and for every pair* $(g_1, g_2) \in \mathcal{W}_w^i$ *there exists the epimorfism* $F_2/\sqrt[G]{w}^{\,i} \to \langle g_1, g_2 \rangle$*.*
ii. The homomorphism $F_2/\sqrt[G]{w} \xrightarrow{\prod_{i=1}^m \lambda_i} \prod_{i=1}^m F_2/\sqrt[G]{w}^{\,i}$*, where* $\lambda_i : F_2/\sqrt[G]{w} \to F_n/\sqrt[G]{w}^{\,i}$ *is the natural epimorphism, is an injection; a faithful representation* $\rho : F_2/\sqrt[G]{w} \to G$ *exists if and only if the homomorphism* $\lambda_i : F_2/\sqrt[G]{w} \to F_2/\sqrt[G]{w}^{\,i}$ *is an isomorphism for some* i*.*

Example.



The projectivization.

Let

$$w(x,y) = x^{c_1}y^{d_1}x^{c_2}y^{d_2}\cdots x^{c_k}y^{d_k}\cdots x^{c_r}y^{d_r},$$

$$\mathcal{X}_w = \begin{pmatrix}\mathcal{X}_{11} & \mathcal{X}_{12}\\ \mathcal{X}_{21} & \mathcal{X}_{22}\end{pmatrix} = \widetilde{w}^*(X,Y) = \hat{X}^{c_1}\hat{Y}^{d_1}\hat{X}^{c_2}\hat{Y}^{d_2}\cdots\hat{X}^{c_k}\hat{Y}^{d_k}\cdots\hat{X}^{c_r}\hat{Y}^{d_r}$$

where

$$\hat{X}^{c_i} = \begin{cases} X^{c_i} & \text{if } c_i > 0,\\ \left(X^*\right)^{|c_i|} & \text{if } c_i < 0 \end{cases} \qquad \hat{Y}^{d_i} = \begin{cases} Y^{d_i} & \text{if } d_i > 0,\\ \left(Y^*\right)^{|d_i|} & \text{if } d_i < 0. \end{cases}$$

$$c^- = \sum_{c_j<0} \mid c_j \mid \quad d^- = \sum_{d_j<0} \mid d_j \mid .$$

Theorem.
Let $w \in [F_2, F_2]$ and let $\alpha \in \mathbb{C}$. Then
i. there exists a pair $(g_1, g_2) \in \mathrm{M}_2^2(\mathbb{C})$ such that

$$\mathcal{X}_{11}(g_1,g_2) = 0,\ \ \mathcal{X}_{22}(g_1,g_2) - \alpha\big(det g_1\big)^{c^-}\big(\det g_2\big)^{d^-} = 0$$

where $\mathcal{X}_{pq}(g_1,g_2)$ is the (pq) entry of the matrix $\widetilde{w}^*(g_1,g_2) \in \mathrm{M}_2(\mathbb{C})$;
ii. if $(g_1,g_2) \in \mathrm{GL}_2(\mathbb{C})^2$ where (g_1,g_2) is a pair which satisfies the condition i., then there exists a non-central element $g \in \mathrm{Im}\ \widetilde{w}$ such that $\mathrm{tr}\, g = \alpha$.