

Uniform Stability of High-Rank Lattices

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§1. Rigidity and Stability

- G - (semi) simple Lie group over a local field K .
- $\Gamma \leq G$ - a lattice (discrete subgroup of finite covolume; uniform (=cocompact) or non-uniform).
- $rk(G) = rank_K(G)$
- “usually” a difference between $rk = 1$ and $rk \geq 2$ (= high rank), e.g., local rigidity, strong rigidity, super-rigidity, congruence subgroup problem.

$$SL_2(\mathbb{Z}) \text{ versus } SL_n(\mathbb{Z}), \geq 3$$

This talk is about Ulam Stability which is another such property.

A typical rigidity result: There is a clear, easy to understand, family of representations of Γ . The theorem says that if something is “similar” it is already in the family.

Ulam stability is such a statement for **“almost representations”**

More formally:

Let Γ be a group and $\mathfrak{g} = (G_n, d_n)$ -family of groups with $d_n = \text{bi invariant metric}$.

Def: Γ is **uniform/Ulam stable** w.r.t. \mathfrak{g} , if $\forall \varepsilon > 0, \exists \delta > 0$, such that

$\forall n, \forall \text{map } \varphi : \Gamma \rightarrow G_n$ with

$$d_n(\varphi(gh), \varphi(g)\varphi(h)) \leq \delta, \forall g, h \in \Gamma \quad (*)$$

\exists a **homomorphism** $\psi : \Gamma \rightarrow G_n$

s.t.

$$d_n(\varphi(g), \psi(g)) \leq \varepsilon, \forall g \in \Gamma \quad (**)$$

i.e., every “almost representation” is just a small deformation of a true representation.

Warning: Do not confuse with (ordinary) **stability** which is equivalent to:

If $\varphi_n : \Gamma \rightarrow G_n$ s.t. $\forall g, h \in \Gamma \quad d_n(\varphi_n(gh), \varphi_n(g)\varphi_n(h)) \xrightarrow{n \rightarrow \infty} 0$

then $\exists \psi_n : \Gamma \rightarrow G_n$ homomorphisms with

$$d_n(\varphi_n(g), \psi_n(g)) \xrightarrow{n \rightarrow \infty} 0 \quad \forall g \in \Gamma$$

Uniform Stability is equivalent to

If $\varphi_n : \Gamma \rightarrow G_n$ s.t. $\sup_{g, h \in \Gamma} d_n(\varphi_n(gh), \varphi_n(g)\varphi_n(h)) \xrightarrow{n \rightarrow \infty} 0$

then $\exists \psi_n : \Gamma \rightarrow G_n$ homomorphisms with

$$\sup_g d_n(\varphi_n(g), \psi_n(g)) \xrightarrow{n \rightarrow \infty} 0$$

Today we will work only with $G_n = U(n)$ and d_n - metric induced by a **submultiplicative** norm $\|\cdot\|$ on $M_n(\mathbb{C})$, i.e. $\|AB\| \leq \|A\| \|B\|$ and $d_n(A, B) = \|A - B\|$.

Ex: (a) the operator norm $\|\cdot\|_\infty = \|\cdot\|_{op}$

(b) The Frobenius norm = L^2 -norm

(c) The p -Schatten norm ($1 \leq p < \infty$) $\|A\|_p = (\text{tr}|A|^p)^{1/p}$

when $|A| = \sqrt{A * A}$ (So (b) is the case $p = 2$ of (c))

Non-example: The Hilbert-Schmidt norm

$$\|A\|_{HS} = (\text{tr} \frac{1}{n} |A|^2)^{1/2} = \frac{1}{\sqrt{n}} \|A\|_2$$

Theorem (Glebsky-Lubotzky-Monod-Rangarajan)

$\mathfrak{g} = (U(n), d_n)$, d_n -submultiplicative, Γ - a lattice, G - a high rank simple Lie group over a local field K . Then Γ is Ulam-stable provided G satisfies condition $G(Q_1, Q_2)$.

Remark: $G(Q_1, Q_2)$ to be defined later, is satisfied by “most” simple groups. E.g., always if K is non-archimedean, **or** for $SL_d(\mathbb{R})$ if $d \geq 4$ and $SL_d(\mathbb{C})$, $d \geq 3$.
As of now: we do not know for $G = SL_3(\mathbb{R})$.

§. 2 History

Theorem (Kazhdan 1982)

Amenable groups are **strongly** Ulam stable w.r.t. the operator norm.

Strongly means even w.r.t. infinite dimensional Hilbert space.

Theorem (Burger-Ozawa-Thom 2013)

If Γ any discrete group containing a free non-abelian subgroup, then Γ is not strongly Ulam stable.

Pf. Free groups are **not** Ulam stable and induce the “almost rep”. \square

Open problem: Does “strong Ulam stability” characterize amenability?

Theorem (Burger-Ozawa-Thom 2013)

If $H_b^2(\Gamma, \mathbb{R}) \rightarrow H^2(\Gamma, \mathbb{R})$ is not injective, then Γ does not have Ulam-Stability.

Cor: Lattices in rank one groups are not Ulam stable.

Theorem (Burger-Ozawa-Thom 2013)

$SL_d(\mathbb{Z})$ (or more generally $SL_d(\mathcal{O}_S)$) are Ulam stable for $d \geq 3$.

Remark: The proof used “bounded generation” of these groups.

According to a recent result of **Corvaja, Rapinchuk, Ren** and **Zannier**, cocompact lattices are never boundedly generated.

Recall: (i) $H_b^n(\Gamma, V) = 0 \forall n$ when Γ is amenable.

(ii) $H_b^n(\Gamma, V) = 0 \forall n$ when Γ is high-rank lattice and $V^\Gamma = \{0\}$.

- **Burger-Monod** for $n = 2$
- **Burger-Shalom** for $n = 2$, more general V , different pf
- **Monod** all n .

This led **Monod** already in his ICM talk (2000) to ask:

Is there a connection between Ulam stability and bounded cohomology?

Answer: Yes, but . . .

§3 Stability and 2nd cohomology

There has been quite a lot of progress on ordinary stability in recent years. A major technical tool to prove stability is the following (\mathfrak{u} = ultrafilter on \mathbb{N}):

Theorem (de Chiffre, Glebsky, Lubotzky, Thom)

Let $\mathcal{L}_n = \text{Lie}(U(n))$ with the norms as before and $\mathcal{L} = \prod_{\mathfrak{u}} \mathcal{L}_n$ the topological ultra product of \mathcal{L}_n . The “almost homomorphisms” φ_n (in the standard sense) define an action of Γ on \mathcal{L} . If $H^2(\Gamma, \mathcal{L}) = 0$ then $\{\varphi_n\}$ are close to homo's ψ_n . In particular, if we work for example with the Frobenius norm and $H^2(\Gamma, V) = 0$ for every Hilbert space, then Γ is Frobenius stable.

What does H^2 have to do with stability?

The maps $\varphi_n : \Gamma \rightarrow U(n)$ give rise to true homo: φ^* :

$$\begin{array}{ccc} & & \Pi U(n) \\ & \nearrow \psi & \downarrow \\ \Gamma & \xrightarrow{\varphi^*} & \Pi U(n)/inf \end{array}$$

Γ is stable iff this φ^* can be lifted to *homomorphism* $\psi : \Gamma \rightarrow \Pi U(n)$.

The kernel K is a “very” non-commutative group, but if **the norm is submultiplicative** K can be approximated by abelian small steps.

Vanishing of H^2 gives small extensions and the limit gives ψ .

The same strategy can, in principle, work for the uniform stability. BUT

(I) The relevant cohomology here is the bounded cohomology!

(II) We need ψ to be **internal** i.e., $\psi = (\psi_n)_{n \in \mathbb{N}}$.

(In the ordinary stability we care about the values of ψ only on the generators; every ψ is internal).

Putting points (I) and (II) together gives:

Proposition

The group Γ is Ulam-g-stable if and only if every homomorphism φ **that has an internal lift**

$$\begin{array}{ccc} & & \Pi U(n) \\ & \nearrow \psi & \downarrow \\ * \Gamma & \xrightarrow{\varphi} & \Pi U(n)/inf \end{array}$$

has an **internal** lift homomorphism ψ .

What captures this lifting problem, when $\| \cdot \|$ is **submultiplicative**, is a **new** cohomology theory, which we call the **asymptotic cohomology** $H_a^n(\Gamma, \mathcal{L})$ of Γ (really of $*\Gamma$) which deals only with **internal** cochains.

Main technical theorem:

Theorem (Glebsky, Lubotzky, Monod, Ramanujan)

If $H_a^2(\Gamma, \mathcal{L}) = 0$ (w.r.t. to φ obtained from almost homo in the uniform sense - φ_n) then $\{\varphi_n\}$ are near true homo's $\{\psi_n\}$.

Remarks:

1) There is a canonical map

$$H_a^n(\Gamma, \mathcal{L}) \longrightarrow H_b^n(\Gamma, \mathcal{L})$$

but we do not know if this is injective(?) surjective(?)

2) $H_a^n(\Gamma, V) = 0 \forall n, \forall V$ if Γ is amenable, so we recover **Kazhdan** Thm.

Our goal: Prove $H_a^2(\Gamma, \mathcal{L}) = 0$ when Γ is high-rank lattice

Step one: Shapiro Induction

Theorem

If Γ high-rank lattice in G , then \forall dual Banach space V with Γ action:

$$H_b^2(\Gamma, V) = H_b^2(G, W)$$

where $W = \text{Ind}_{\Gamma}^G(V)$.

By **a lot** of work, we can imitate an **Eckman-Shapiro** approach to get

$$H_a^2(\Gamma, \mathcal{L}) = H_a^2(G, \mathcal{W})$$

for $\mathcal{W} = \text{Ind}_{\Gamma}^G(\mathcal{L})$.

Why a lot of work? Even a standard result like Shapiro Lemma requires for $*\Gamma$ in $*G$ (e.g. what is $L^\infty(*\Gamma \backslash *G)$? what is $L^2(*\Gamma \backslash *G)$?).

Step two: Reducing to trivial module

In the **Monod-Shalom** proof, they obtain

Theorem

For dual separable Banach space W with G action:

$$H_b^2(G, W) = H_b^2(G, W^G)$$

In particular, if $V^\Gamma = W^G = 0$, then $H_b^2(G, W) = 0$.

A similar conclusion holds in our setting too:

Theorem

Suppose \mathcal{W} has no fixed point "upto infinitesimals". Then

$$H_a^2(G, \mathcal{W}) = 0$$

Step three: Dealing with Fixed Points

We can do it for all lattices Γ in a simple Lie group G if G satisfies $G(Q_1, Q_2)$.

$G(Q_1, Q_2)$ **means:** G has two parabolic subgroups Q_1 and Q_2 satisfying:

(i) $Q_1 \cap Q_2$ contains a minimal parabolic P , and $\langle Q_1, Q_2 \rangle = G$

(ii) $H_b^2(Q_i, \mathbb{R}) = 0$ and

$H_b^3(Q_i, \mathbb{R})$ is Hausdorff.

(Note: conditions on the bounded cohomology with trivial coefficients).

“Most” simple groups satisfy $G(Q_1, Q_2)$, so our result, as of now, is quite general but still not complete.