Algebras of permutations

Sergei Lando

National Research University Higher School of Economics,
Skolkovo Institute of Science and Technology,
Moscow, Russia

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Enumerating ramified coverings of the sphere

In the COMPLEX CASE, one fixes m points in $\mathbb{C}P^1$, and ramification profile over each point. The *Hurwitz problem* consists in enumerating ramified coverings of the sphere having the specified ramification profile over each of the prescribed points.

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In the REAL CASE, one requires in addition that the configuration of the ramification points is symmetric with respect to the complex conjugation, taking ramification profiles into account. The *real Hurwitz problem* consists in enumerating real ramified coverings of the sphere having the specified ramification profile over each of the prescribed points.

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- complex Hurwitz numbers enumerate decompositions of a given permutation into a product of given number of permutations of prescribed cyclic types; in other words, they are *connection coefficients* in the centers $Z\mathbb{C}[\mathbb{S}_d]$ of the group algebras of symmetric groups:

$$h_{\lambda_1,\dots,\lambda_m} = \frac{1}{d!}\#\{(\sigma_1,\dots,\sigma_m)|\sigma_1\circ\dots\circ\sigma_m = \mathrm{id}, \sigma_1 \in C_{\lambda_1},\dots,\sigma_m \in C_{\lambda_m}\}$$

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Here $\lambda_1,\dots,\lambda_m$ are partitions of d, the degree of the covering; such a partition determines a cyclic type of permutations.
In the center $Z\mathbb{C}[\mathbb{S}_d]$ of the group algebra of $\mathbb{S}(d)$, we have

$$h_{\lambda_1,\dots,\lambda_m} = [C_{1^d}]C_{\lambda_1}C_{\lambda_2}\dots C_{\lambda_m}.$$

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- for polynomials of degree n, this number is n^{n-1}, (this result is known as the *Cayley formula*) the exponential generating function being the *Lambert function*

$$L(q) = \sum_{n=1}^{\infty} n^{n-1}\frac{q^n}{n!}; \quad q = L(q)e^{-L(q)};$$

Generating functions for Hurwitz numbers

Define exponential generating functions in a variable u (recording the number of transpositions) and infinitely many variables $p_1, p_2, \dots$ (recording the parts of the partitions):

$$H^\circ(u; p_1, p_2, \dots) = \sum_{m=0}^{\infty} \sum_{\mu} h^\circ_{m;\mu} p_{\mu_1} p_{\mu_2} \dots \frac{u^m}{m!};$$

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The two are related by

$$H^\circ = \exp(H).$$

Series expansion for Hurwitz numbers

It is clear from the definition that the coefficients of both H and H° are rational. The first terms of the power series expansions are

$$\begin{aligned}
H^\circ(u; p_1, \dots) &= 1 + p_1 + \frac{p_2 u}{2} + \frac{p_1^2}{2} + \frac{p_3 u^2}{2} + \frac{1}{2} p_2 p_1 u + \frac{p_1^3}{6} + \frac{p_2 u^3}{12} \\
&\quad + \frac{2 p_4 u^3}{3} + \frac{1}{4} p_1^2 u^2 + \frac{1}{2} p_3 p_1 u^2 + \frac{1}{8} p_2^2 u^2 + \frac{1}{4} p_2 p_1^2 u + \frac{p_1^4}{24} + \cdots \\
H(u; p_1, \dots) &= p_1 + \frac{p_2 u}{2} + \left(\frac{p_1^2}{4} + \frac{p_3}{2} \right) u^2 + \left(\frac{2 p_1 p_2}{3} + \frac{p_2}{12} + \frac{2 p_4}{3} \right) u^3 \\
&\quad + \left(\frac{p_1^3}{6} + \frac{p_1^2}{48} + \frac{9 p_3 p_1}{8} + \frac{p_2^2}{2} + \frac{3 p_3}{8} + \frac{25 p_5}{24} \right) u^4 + \cdots
\end{aligned}$$

Complex case

Complex case

- the generating function $H^\circ(u; p_1, p_2, \dots)$ for simple complex Hurwitz numbers satisfies the so-called *cut-&-join* equation (I. Goulden and D. Jackson, 1996)

$$\frac{\partial H^\circ}{\partial u} = \sum_{i,j=1}^{\infty} \left(ij p_{i+j} \frac{\partial^2 H^\circ}{\partial p_i \partial p_j} + (i+j) p_i p_j \frac{\partial H^\circ}{\partial p_{i+j}} \right).$$

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  Schur polynomials are the eigenvectors of the cut-&-join differential operator on the right, and the corresponding eigenvalues are known rational numbers;

Complex case

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- the generating function $H^\circ(u; p_1, p_2, \dots)$ is a τ-function of the Kadomtsev–Petviashvili integrable hierarchy of partial differential equations (Okounkov, 2002), in particular:

$$\frac{\partial^2 H}{\partial p_2^2} = \frac{\partial^2 H}{\partial p_1 \partial p_3} - \frac{1}{2}\left(\frac{\partial^2 H}{\partial p_1^2}\right)^2 - \frac{1}{12}\frac{\partial^4 H}{\partial p_1^4};$$

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- simple complex Hurwitz numbers are related to the geometry of moduli spaces of algebraic curves through the ELSV-formula (1999).

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Definition

A *state* is a splitting of a finite set $D = D^+ \sqcup D^-$ into a disjoint union of one- and two-element subsets such that each two-element subset contains an element from D^+ and an element from D^-. A *transition* is an ordered pair of states.

Real case

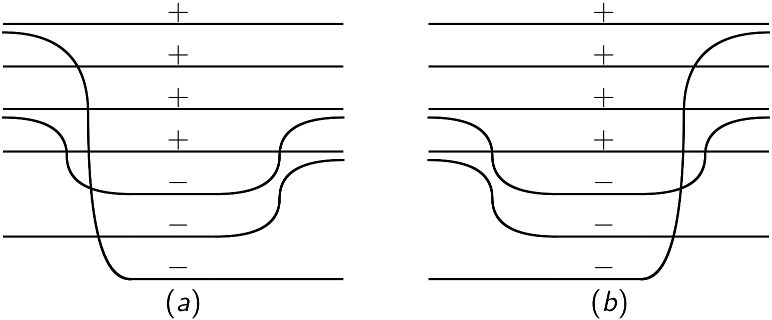

Figure: (a) A transition on a set $D = D^+ \sqcup D^-$, $|D^+| = d^+ = 4$, $|D^-| = d^- = 3$. (b) The adjacent transition.

Real case

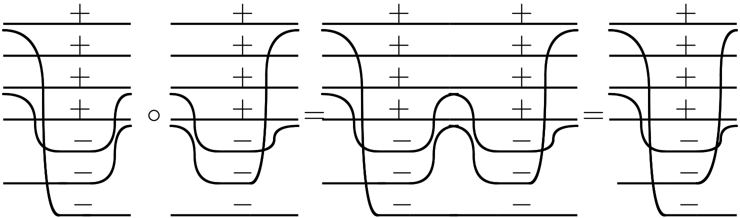

Figure: A nonzero product of two transitions

Algebras of transitions

Definition

Let T_{d^+,d^-} denote the algebra of transitions spanned by all transitions on a given set $D = D^+ \bigsqcup D^-$, $|D^\pm| = d^\pm$, endowed with the composition product. Let $A_{d^+,d^-} = T_{d^+,d^-}^{\mathbb{S}(d^+)\times\mathbb{S}(d^-)}$ denote the $\mathbb{S}(d^+) \times \mathbb{S}(d^-)$-invariant part of T_{d^+,d^-}.

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The algebra T_{d^+,d^-} serves as an analogue of the group algebra $\mathbb{C}[\mathbb{S}(d)]$, and its subalgebra A_{d^+,d^-} is an analogue of the center $Z\mathbb{C}[\mathbb{S}(d)]$. In contrast to the latter, it is noncommutative, just associative.

Real case

Real case

- the exponential generating function for numbers enumerating polynomials has the form

$$\mathrm{tg}(q) + \frac{1}{\cos(q)}.$$

Simple framed real Hurwitz numbers

(M. Kazarian, S. L., S. Natanzon, 2021)
The generating function for simple framed real Hurwitz numbers (those with at most one degenerate branch point)

$$H^{\mathbb{R}}(u; p_1^+, p_2^+, \dots; p_1^-, p_2^-, \dots; q_1, q_2, \dots)$$

satisfies the cut-&-join equation

$$\frac{\partial H^{\mathbb{R}}}{\partial u} = W^+(H^{\mathbb{R}}),$$

where the linear partial differential operator W^+ is defined as follows:

$$W^+ = \sum_{i,j=1}^{\infty} \left(p_i^{\bar{i}} p_j^+ \frac{\partial}{\partial p_{i+j}^{\bar{i}}} + p_{i+j}^{\bar{i}} \frac{\partial^2}{\partial p_i^{\bar{i}} \partial p_j^+} \right) + \sum_{i=1}^{\infty} \left(i p_{2i}^+ \frac{\partial}{\partial q_i} + q_i \frac{\partial}{\partial p_{2i}^+} \right),$$

where $\bar{i}$, for a positive integer i, denotes the sign $+$, provided i is even, and the sign $-$ otherwise.

Comparing complex and real cases

The cut-&-join operator W in the complex case (which is, essentially the Calogero–Moser operator) splits into a direct sum of homogeneous finite dimensional linear operators. Each of these operators is diagonalized in the basis of Schur polynomials, with known rational eigenvalues.

Comparing complex and real cases

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The cut-&-join operator W^+ in the real case splits into a direct sum of homogeneous finite dimensional linear operators. These operators are diagonalizable, but in contrast to the complex case the eigenvalues are not rational. There is an efficient recurrence for computing these operators (Krasilnikov, 2022).

Polynomial real Hurwitz numbers

In the polynomial case (ramified coverings of $\mathbb{C}P^1$ by $\mathbb{C}P^1$ having a point with a single preimage), with all real critical values, I. Itenberg and D. Zvonkine (2018) managed to assign a sign (either $+$ or $-$) to each polynomial so that the algebraic number of polynomial coverings counted with this sign becomes independent of the order of the critical values.

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The sign is defined as the product of the signs of critical values, while the sign of a critical value t is the number of disorders in the degrees of the preimages of t. A *disorder* is a pair of preimages such that the degree of a smaller preimage is greater than that of a larger one.

Generating functions for complex Hurwitz numbers

Fix a tuple $\Lambda = (\lambda_1, \dots, \lambda_k)$ of partitions and denote by $H_\Lambda(q)$ the exponential generating function enumerating ramified coverings of the sphere having k points of degenerate ramification with profiles $\lambda_1, \dots, \lambda_k$, and m additional points of simple ramification. Then

$$H_\Lambda(q) = P_\Lambda(q, L(q), L'(q)),$$

for some polynomial P_Λ in three variable.

Polynomial real Hurwitz numbers

Fix a tuple $\Lambda = (\lambda_1, \dots, \lambda_k)$ of partitions and denote by $S_\Lambda(q)$ the exponential generating function enumerating real polynomial ramified coverings of the sphere having k points of degenerate ramification with profiles $\lambda_1, \dots, \lambda_k$, and m additional points of simple ramification, counted with signs. Then

$$S_\Lambda(q) = P_\Lambda(q, \mathrm{tg}(q)) + \frac{1}{\cos(q)} Q_\Lambda(q, \mathrm{tg}(q)),$$

for some polynomials P_Λ and Q_Λ in two variables.

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Problem. How can one extend the notion of a real covering to meromorphic functions on real algebraic curves that are more general than polynomial?

Algebras of permutations in theory of finite type knot invariants

The theory of finite type knot invariants associates to a knot invariant of order at most n a function on chord diagrams with n chords. Such a function can be considered as a function on arc diagrams that is invariant with respect to cyclic shifts. In turn, an arc diagram is a permutation of special kind, an involution without fixed point.

Algebras of permutations in theory of finite type knot invariants

One of the main problems in the study of weight systems is developing ways of efficient computations. This is, in particular, an essential step in understanding weight systems associated to Lie algebras (quantum knot invariants). A recent idea of Kazarian provides such a tool for the series of Lie algebras $\mathfrak{gl}(N)$, $N = 1, 2, 3, \ldots$. It consists in extending the $\mathfrak{gl}(N)$-weight systems to arbitrary permutations, not necessarily involutions without fixed points. It happens that one can get recurrence relations expressing the value of the $\mathfrak{gl}(N)$ weight system on a permutation in terms of its values in simpler permutations.

Algebras of permutations in theory of finite type knot invariants

The extended $\mathfrak{gl}(N)$ weight systems are invariant with respect to cyclic shifts of permutations. They also are multiplicative with respect to *concatenation product* of permutations. This leads to the following algebra. A permutation is said to be *connected* if none of its cyclic shifts is a concatenation product of two permutations, both of smaller orders. The algebra $\mathcal{A}$ is freely generated by cyclic equivalence classes of connected permutations.

Algebras of permutations in theory of finite type knot invariants

The algebra $\mathcal{A}$ is endowed with a natural comultiplication:

$$\mu : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}; \qquad \mu : \alpha \mapsto \sum_{I \sqcup J = \{1,2,\dots,m\}} \alpha|_I \otimes \alpha|_J, \qquad \alpha \in \mathbb{S}_m,$$

which makes it into a graded Hopf algebra.

Thank you
for your attention