On Gruenberg-Kegel graphs and beyond

Natalia V. Maslova

Abstract. The Gruenberg–Kegel graph (or the prime graph) of a finite group G is a simple graph whose vertices are the prime divisors of $|G|$, with primes p and q adjacent in this graph if and only if pq is an element order of G. The concept of Gruenberg–Kegel graph proved to be very useful in finite group theory and in algebraic combinatorics as well as with connection to research of some cohomological questions in integral group rings. In this talk, we discuss recent results on characterization of finite groups by Gruenberg-Kegel graph and by isomorphism type of Gruenberg-Kegel graph as well as combinatorial properties of Gruenberg–Kegel graphs.

This talk is dedicated to the bright memory of Prof. Nikolai A. Vavilov and Prof. Otto H. Kegel.

Let Ω be a finite set of positive integers. Define $\pi(\Omega)$ to be the set of all prime divisors of integers from Ω. A simple graph $\Gamma(\Omega)$ whose vertex set is $\pi(\Omega)$ and two distinct vertices p and q are adjacent in this graph if and only if pq divides some element from Ω is called the *prime graph* of Ω.

Let G be a finite group. Denote by $\omega(G)$ the *spectrum* of G, that is, the set of all its element orders. The *Gruenberg-Kegel graph* (or the *prime graph*) $\Gamma(G)$ is a simple graph whose vertices are the prime divisors of $|G|$, with primes p and q adjacent in this graph if and only if pq is an element order of G. It is clear that $\Gamma(G) = \Gamma(\omega(G))$.

The concept of Gruenberg–Kegel graph appeared in the unpublished manuscript [5] by K. Gruenberg and O. Kegel, and proved to be very useful in finite group theory and in algebraic combinatorics as well as with connection to research of some cohomological questions in integral group rings.

In this talk, we discuss the following:

- the question of coincidence of Gruenberg–Kegel graphs of non-isomorphic finite groups;
- recent results on characterization of finite groups by Gruenberg–Kegel graph and by isomorphism type of Gruenberg–Kegel graph;

- the mutual impact of properties of Gruenberg–Kegel graphs and of graphs defined on groups (for example, the commuting graph, the power graph, the enhanced power graph, the subnormality graph);
- combinatorial properties of Gruenberg–Kegel graphs, in particular, we answer the following question: What are the strongly regular graphs which are isomorphic to Gruenberg–Kegel graphs of finite groups?

This talk is based on joint projects with V. Arvind, P. J. Cameron, M.-Zh. Chen, I. B. Gorshkov, A. Lucchini, X. Ma, L. G. Nechitailo, N. Otmen, V. V. Panshin, A. M. Staroletov, N. Yang, and M. R. Zinov'eva.

Some discussed results as well as surveys of parts of the research area can be found in [1, 2, 3, 4, 6, 7].

Acknowledgements. The speaker is partially supported by the Ministry of Science and Higher Education of the Russian Federation, project no. 075-02-2025-1549 for the development of the regional scientific and educational mathematical center "Ural Mathematical Center" (for example, recent results about characterization of a finite group by isomorphism type of Gruenberg–Kegel graph) and by the Russian Science Foundation, project 24-11-00119, https://rscf.ru/en/project/24-11-00119/ (for example, results about characterization by Gruenberg–Kegel graph of the groups $F_4(q)$ with q odd and the finite simple groups with orders having prime divisors at most 13).

References

[1] V. Arvind, P. J. Cameron, X. Ma, N. V. Maslova, Aspects of the commuting graph, *J. Algebra*, available at https://doi.org/10.1016/j.jalgebra.2025.07.020 (2025).

[2] P. J. Cameron, N. V. Maslova, Criterion of unrecognizability of a finite group by its Gruenberg-Kegel graph. *J. Algebra*, **607**: Part A (2022), 186–213.

[3] M.-Zh. Chen, N. V. Maslova, M. R. Zinov'eva, On characterization of groups by isomorphism type of Gruenberg-Kegel graph, preprint (2025), available at arXiv:2504.14703 [math.GR] (2025).

[4] M.-Zh. Chen, I. B. Gorshkov, N. V. Maslova, N. Yang, On combinatorial properties of Gruenberg–Kegel graphs of finite groups. *Monatshefte für Mathematik*, **205** (2024), 711–723.

[5] K. W. Gruenberg, O. Kegel, Unpublished manuscript (1975).

[6] N. V. Maslova, On arithmetical properties and arithmetical characterizations of finite groups, in: *Proceedings of* $(WM)^2$ *— World Meeting for Women in Mathematics* — 2022, to appear, available at arXiv:2401.04633 [math.GR] (2024).

[7] N. V. Maslova, V. V. Panshin, A. M. Staroletov, On characterization by Gruenberg-Kegel graph of finite simple exceptional groups of Lie type. *European J. Math.*, **9** (2023), Article number: 78.

Natalia V. Maslova
N.N. Krasovskii Institute of Mathematics and Mechanics UB RAS
Yekaterinburg, Russia
Ural Mathematical Center
Yekaterinburg, Russia
S.L. Sobolev Institute of Mathematics SB RAS
Novosibirsk, Russia
e-mail: butterson@mail.ru