Locally isotropic elementary groups and their central extensions

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Part I. General linear groups

Elementary transvections

Let R be a unital commutative ring.

Definition

An *elementary transvection* in $\mathrm{GL}(n, R)$ is $t_{ij}(r) = 1 + e_{ij} r$ for some $1 \leq i \neq j \leq n$ and $r \in R$.

They satisfy the *Steinberg relations*

- $t_{ij}(r)\, t_{ij}(s) = t_{ij}(r + s)$;
- $[t_{ij}(r), t_{jk}(s)] = t_{ik}(rs)$ for $i \neq k$;
- $[t_{ij}(r), t_{kl}(s)] = 1$ for $j \neq k$ and $i \neq l$.

Definition

The *elementary subgroup* is

$$\mathrm{E}(n, R) = \langle t_{ij}(r) \mid 1 \leq i \neq j \leq n,\ r \in R \rangle \leq \mathrm{GL}(n, R).$$

Properties of elementary subgroups

Theorem (Suslin'77, Bak'89)

If $n \geq 3$, *then* $\mathrm{E}(n, R) \trianglelefteq \mathrm{GL}(n, R)$ *is normal and perfect.*

If in addition R *is finite dimensional, then* $\mathrm{GL}(n, R)/\,\mathrm{E}(n, R)$ *is solvable and* $\mathrm{E}(n, R)$ *is the largest perfect subgroup.*

Theorem (Wilson'72, Golubchik'73)

Suppose that $n \geq 3$. *A subgroup* $H \trianglelefteq \mathrm{GL}(n, R)$ *is normalized by* $\mathrm{E}(n, R)$ *if and only if*

$$\mathrm{E}(n, R, I) \leq I \leq \mathrm{C}(n, R, I),$$

where

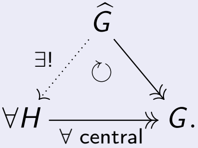
$$\mathrm{E}(n, R, I) = {}^{\mathrm{E}(n,R)}\langle t_{ij}(a) \mid 1 \leq i \neq j \leq n,\ a \in I \rangle,$$
$$\mathrm{C}(n, R, I) = \{ g \in \mathrm{GL}(n, R) \mid \mathrm{Im}(g) \in \mathrm{GL}(n, R/I) \text{ is central} \}.$$

Perfect central extensions

Let $G = [G, G]$ be a perfect group.

Definition

A central extension $\widehat{G} \twoheadrightarrow G$ is *universal* if



The *Schur multiplier* of G is $\mathrm{M}(G) = \mathrm{H}_2(G) = \mathrm{Ker}(\widehat{G} \to G)$.

Theorem

A universal central extension $\widehat{G} \twoheadrightarrow G$ exists, it is perfect and unique up to unique isomorphism. If $H \twoheadrightarrow G$ is a perfect central extension, then $\widehat{H} \cong \widehat{G}$.

Steinberg groups

Definition

The *Steinberg group* $\mathrm{St}(n, R)$ for $n \geq 3$ is given by generators $x_{ij}(r)$, $1 \leq i \neq j \leq n$, $r \in R$ and the Steinberg relations.

Theorem (van der Kallen'76, van der Kallen–Stein'77)

The Steinberg group is perfect for $n \geq 3$. If $n \geq 4$ (or $n = 3$ and R is semi-local), then it is a central extension of $\mathrm{E}(n, R)$, universal for $n \geq 5$;

$$\mathrm{M}(\mathrm{St}(4, R)) \cong R/\langle 2, (r^2 - r)(s^2 - s)\rangle.$$

Actually, $\widehat{\mathrm{St}}(4, R)$ is generated by $x_{ij}^k(r)$ for $1 \leq i \neq j \neq k \leq 4$, $r \in R$ and by $c(r)$ for $r \in R/\langle 2, (s^2 - s)(t^2 - t)\rangle$ such that for $\{i, j, k, l\} = \{1, 2, 3, 4\}$

c is central; $\quad$ c and x_{ij}^k are homomorphisms;

$$x_{ij}^k(r) = x_{ij}^l(r)\, c(r); \qquad [x_{ik}^u(r), x_{jk}^v(s)] = [x_{ki}^u(r), x_{kj}^v(s)] = 1;$$

$$[x_{ij}^u(r), x_{kl}^v(s)] = c(rs); \qquad [x_{ij}^u(r), x_{jk}^v(s)] = x_{ik}^j(rs)\, c(r(s^2 - s)).$$

Steinberg symbols

The kernel of $\mathsf{St}(n, R) \to \mathsf{E}(n, R)$ can be explicitly calculated by generators and relations if R is semi-local, of stable rank 1, or a Dedekind domain of arithmetic type under additional assumptions (e.g. if $R = \mathbb{Z}$).

Theorem (Matsumoto'69)

If R is a field and $n \geq 3$, then $\mathsf{E}(n, R) = \mathrm{SL}(n, R)$ and

$$\mathrm{Ker}\big(\mathsf{St}(n, R) \to \mathsf{E}(n, R)\big) \cong (R^* \otimes_{\mathbb{Z}} R^*)/\langle u \otimes (1-u) \mid u \neq 0, 1\rangle.$$

Namely, let $w_{ij}(u) = x_{ij}(u)\, x_{ji}(-u^{-1})\, x_{ij}(u)$ and $h_{ij}(u) = w_{ij}(u)\, w_{ij}(-1)$ for $u \neq 0$. The *Steinberg symbols*

$$\{u, v\} = h_{ij}(uv)\, h_{ij}(u)^{-1}\, h_{ij}(v)^{-1}$$

are independent of the indices, generate the kernel and satisfy

$$\{uv, w\} = \{u, w\}\,\{v, w\}; \qquad \{u, 1-u\} = 1 \text{ for } u \neq 0, 1;$$
$$\{u, vw\} = \{u, v\}\,\{u, w\}; \qquad \{u, -u\} = 1.$$

Part II. Locally isotropic linear groups

Locally isotropic general linear groups

Now let P be a finite projective R-module of rank at least 3 at every maximal ideal.

The goal is to construct analogues $\mathrm{E}(P, R) \leq \mathrm{Aut}(P)$ and $\mathrm{St}(P, R)$ of the elementary group and the Steinberg group.

Example

Let $R = \mathbb{R}[x_0, \ldots, x_n]/(x_0^2 + \ldots + x_n^2 - 1)$ be the ring of rational functions on $\mathbb{S}^n$. If n is even, then the tangent bundle

$$P = \left\{ \vec{v} \in R^{n+1} \mid \sum_i v_i x_i = 0 \right\}$$

has no unimodular vectors.

The problem is that in general there are no well behaved unipotent subgroups.

Globally isotropic case

If $\mathrm{Aut}(P)$ has a proper parabolic subgroup, its elementary subgroup was constructed by V. Petrov and A. Stavrova.

Suppose that $P = P_1 \oplus P_2$, where P_i have positive rank at every maximal ideal.

Definition (Petrov–Stavrova'09)

The *elementary subgroup* is

$$\mathsf{E}(P, R) = \langle 1 + \mathrm{Hom}(P_1, P_2),\ 1 + \mathrm{Hom}(P_2, P_1) \rangle \leq \mathrm{Aut}(P).$$

The subgroup $\mathsf{E}(P, R)$ is independent of the splitting $P = P_1 \oplus P_2$. This subgroup is perfect and normal, and there is a standard classification of subgroups of $\mathrm{Aut}(P)$ normalized by $\mathsf{E}(P, R)$.

Germs of unipotent subgroups

Return to the general case and suppose for simplicity that R is a domain.

Take $0 \neq s \in R$ such that P_s is free. Choose a basis $(\frac{e_i}{s^m})_{i=1}^n$ and a dual basis $(\frac{f_i}{s^m})_{i=1}^n$ in P_s, where $e_i \in P$ and $f_i \in P^*$.

Crucial observation

The root homomorphism

$$t_{ij} \colon R_s \to \mathrm{Aut}(P_s),\ \frac{r}{s^k} \mapsto 1 + \frac{e_i r f_j}{s^{2m+k}}$$

induces well-defined homomorphisms

$$s^{2m+l} t_{ij} \colon s^{2m+l} R \to \mathrm{Aut}(P),\ s^{2m+l} r \mapsto 1 + e_i s^r r f_j \text{ for } l \geq 0.$$

These $s^{2m+l} t_{ij}$ behave well in the limit $l \to +\infty$.

Co-localization

Definition

The *co-localization* of R at s is the projective limit

$$s^\infty R = \varprojlim_{i \to +\infty} s^i R$$

in a suitable overcategory $\mathcal{C} \supseteq \mathcal{S}et$. The *co-local elementary subgroup*

$$\mathsf{E}(P, s^\infty R) \leq \mathrm{Aut}(P)$$

is generated by the images of $s^\infty t_{ij} \colon s^\infty R \to \mathrm{Aut}(P)$.

We cannot take $\mathcal{C} = \mathcal{S}et$ because usually $(s^\infty R)_{\mathcal{S}et} = \bigcap_i s^i R = 0$.

The object $s^\infty R$ is a commutative ring in $\mathcal{C}$, but without identity. Fortunately, it is *firm*:

$$s^\infty R \otimes_{s^\infty R} s^\infty R \cong s^\infty R.$$

Localization and co-localization

The co-localization is “dual” to the localization

$$R_s = \varinjlim_i \frac{R}{s^i}.$$

It turns out that $s^\infty R$ is a non-unital algebra over R_s, so $t_{ij}(\frac{r}{s^p})$ normalizes all co-local root subgroups $\mathrm{Im}(s^\infty t_{kl})$ unless $k = j$ and $l = i$. The remaining case follows from

$$\mathrm{Im}(s^\infty t_{ji}) = \big[\mathrm{Im}(s^\infty t_{jk}), \mathrm{Im}(s^\infty t_{ki})\big]$$

using the firmness of $s^\infty R$.

Theorem

The co-local elementary subgroup is normalized by

$$\mathrm{GE}(P, R_s) = \mathsf{E}(n, R_s) \cdot \mathrm{D}(n, R_s).$$

Connection with localization-completion

Recall that the *s-adic completion* of R is

$$\widehat{R}^s = \varprojlim_i R/s^i R.$$

In the commutative diagram below the right square is a pullback and both rows are short exact sequences, for a good category $\mathcal{C}$ and the right understanding of all terms.

$$\begin{array}{ccccc} s^\infty R & \rightarrowtail & R & \twoheadrightarrow & \widehat{R}^s \\ \| & & \downarrow & & \downarrow \\ s^\infty R & \rightarrowtail & R_s & \twoheadrightarrow & (\widehat{R}^s)_s. \end{array}$$

In particular, the composite homomorphism $s^\infty R \to R \to R_s$ is a monomorphism and $s^\infty R$ can be identified with an ideal of R_s.

Gluing

Recall that

$$\mathcal{D}(s) = \{s \notin \mathfrak{p} \in \mathrm{Spec}(R)\} \subseteq \mathrm{Spec}(R).$$

Also, $\mathcal{D}(s) \subseteq \bigcup_{i=1}^{N} \mathcal{D}(s_i)$ if and only if $s^l \in \sum_{i=1}^{N} Rs_i$ for some $l \geq 0$.

The next result is *false* for $\mathcal{C} = \mathcal{S}et$.

Theorem

Suppose that $\mathcal{D}(s) = \bigcup_{i=1}^{N} \mathcal{D}(s_i) \subseteq \mathrm{Spec}(R)$. Then $s^\infty R = \sum_i s_i^\infty R$ and

$$\mathrm{E}(P, s^\infty R) = \prod_i \mathrm{E}(P, s_i^\infty R).$$

Corollary

Every $g \in \mathrm{Aut}(P_s)$ normalizes the co-local elementary subgroup $\mathrm{E}(P, s^\infty R)$, so this group is independent of the choices.

Locally isotropic elementary subgroups

Every finite projective module is free locally in Zarisky topology. Choose $s_1, \dots, s_N \in R$ such that P_{s_i} is free and $\mathrm{Spec}(R) = \bigcup_{i=1}^{N} \mathcal{D}(s_i)$, i.e. $R = \sum_{i=1}^{N} Rs_i$.

Definition

The elementary subgroup is

$$\mathrm{E}(P, R) = \prod_{i=1}^{N} \mathrm{E}(P, s_i^{\infty} R).$$

Theorem (V.'24)

The elementary subgroup is independent of all choices, it is normal and perfect. If P decomposes into a direct sum, then $\mathrm{E}(P, R)$ coincides with the elementary group of Petrov and Stavrova. If $P = R^n$ is free, then $\mathrm{E}(P, R) = \mathrm{E}(n, R)$.

Construction of $\mathcal{C}$: required properties

The category $\mathcal{C}$ has to be sufficiently good.

- It contains $\mathcal{S}et$, or at least the objects R, P, and $\mathrm{Aut}(P)$.
- It has nice projective limits, even more nice than in $\mathcal{S}et$.
- It has rich internal logic to work with images of morphisms and to calculate with Steinberg relations.
- It has nice direct limits. We need to construct subgroups generated by subobjects.

Formally, instead of the last two conditions we require that $\mathcal{C}$ is an *infinitary positive*, so its internal logic is *geometric*.

All main results for elementary groups hold for the choice

$$\mathcal{C} = \mathrm{Ind}(\mathrm{Pro}(\mathcal{S}et)).$$

Construction of $\mathcal{C}$: pro-completion

Definition

The (countable) *pro-completion* of a category $\mathcal{A}$ is the universal category $\mathrm{Pro}(\mathcal{A})$ together with a functor $\mathcal{A} \to \mathrm{Pro}(\mathcal{A})$ such that $\mathrm{Pro}(\mathcal{A})$ has all projective limits of inverse sequences.

Objects of $\mathrm{Pro}(\mathcal{A})$ are *formal projective limits*, i.e. diagrams in $\mathcal{A}$ of type

$$\varprojlim_i{}^{\text{formal}}(X_i) = (\ldots \to X_2 \to X_1 \to X_0).$$

Morphisms are given by

$$\mathrm{Pro}(\mathcal{A})\big(\varprojlim_i{}^{\text{formal}}(X_i), \varprojlim_j{}^{\text{formal}}(Y_j)\big) = \varprojlim_j{}^{\mathcal{S}et} \varinjlim_i{}^{\mathcal{S}et} \mathcal{A}(X_i, Y_j).$$

The functor $\mathcal{A} \to \mathrm{Pro}(\mathcal{A})$ is fully faithful. The category $\mathrm{Pro}(\mathcal{S}et)$ has nice logic (it is *coherent*), but no nice direct limits.

Construction of $\mathcal{C}$: ind-completion

The ind-completion $\mathrm{Ind}(\mathcal{A}) = \mathrm{Pro}(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}}$ is dual to the pro-completion.

Definition

The (countable) *ind-completion* of a category $\mathcal{A}$ is the universal category $\mathrm{Ind}(\mathcal{A})$ together with a functor $\mathcal{A} \to \mathrm{Ind}(\mathcal{A})$ such that $\mathrm{Ind}(\mathcal{A})$ has all direct limits of sequences.

Objects of $\mathrm{Ind}(\mathcal{A})$ are *formal direct limits*, i.e. diagrams in $\mathcal{A}$ of type

$$\varinjlim_i{}^{\text{formal}}(X_i) = \big(X_0 \to X_1 \to X_2 \to \ldots\big).$$

Morphisms are given by

$$\mathrm{Ind}(\mathcal{A})\big(\varinjlim_i{}^{\text{formal}}(X_i), \varinjlim_j{}^{\text{formal}}(Y_j)\big) = \varprojlim_i{}^{\mathcal{S}et} \varinjlim_j{}^{\mathcal{S}et} \mathcal{A}(X_i, Y_j).$$

The functor $\mathcal{A} \to \mathrm{Ind}(\mathcal{A})$ is also fully faithful.

Scheme presentability

To sum up, we have a subgroup $\mathrm{E}(P, R) \trianglelefteq \mathrm{Aut}(P)$ inside $\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}et))$.

Theorem (V.'24)

The elementary subgroup is isomorphic to an object of $\mathrm{Ind}(\mathcal{S}et)$.

There is a canonical "evaluation" functor $\mathrm{Ind}(\mathcal{S}et) \to \mathcal{S}et$, so $\mathrm{E}(P, R)$ may also be considered as an ordinary group.

Theorem (V.'24)

There is a scheme morphism $t\colon \mathbb{A}^N \to \mathcal{A}ut(P)$ *over* R *such that* $\mathrm{E}(P, R) = \langle t(R^N) \rangle$ *and this property holds over all* R*-algebras. The ind-structure is given by the filtration*

$$\{1\} \subseteq \{1\} \cup t(R^N) \cup t(R^N)^{-1} \subseteq \left(\{1\} \cup t(R^N) \cup t(R^N)^{-1}\right)^2 \subseteq \dots$$

In particular, $\mathrm{E}(P, R) \to \mathrm{E}(P/PI, R/I)$ is surjective for any ideal $I \trianglelefteq R$.

Set-theoretic definition

It is possible to define $\mathrm{E}(P, R)$ in elementary terms as follows.

Fix $s_1, \ldots, s_N \in R$ such that P_{s_k} are free over R_{s_k} and $\operatorname{Spec}(R) = \bigcup_{k=1}^{N} \mathcal{D}(s_k)$. Also fix bases $(\frac{e_i}{s_k^m})_{i=1}^{n_k}$ and dual bases $(\frac{f_i}{s_k^m})_{i=1}^{n_k}$ in all P_{s_k}. Let $t_{ij}^k \colon R_{s_k} \to \operatorname{Aut}(P_{s_k})$ be the elementary transvections.

Theorem (V.'24)

For all sufficiently large M the group

$$\langle \operatorname{Im}(s_k^{2m+M} t_{ij}^k) \mid 1 \leq i \neq j \leq n_k,\ 1 \leq k \leq N \rangle \leq \operatorname{Aut}(P)$$

coincides with $\mathrm{E}(P, R)$.

Homotopes

Consider the case of arbitrary ring R, i.e. not necessarily a domain. As before, take $s \in R$ such that P_s is free.

Definition

The s-homotope of R is $R^{(s)} = \{r^{(s)} \mid r \in R\}$. It is a non-unital ring with the operations $r^{(s)} + t^{(s)} = (r+t)^{(s)}$ and $r^{(s)} t^{(s)} = (rts)^{(s)}$.

Clearly, $R^{(s)}$ is an algebra over R (and a *ring crossed module* under the homomorphism $R^{(s)} \to R$, $r^{(s)} \mapsto rs$), but not an ideal if s is a zero divisor. Instead of $s^\infty R$ we need

$$R^{(s^\infty)} = \varprojlim_n{}^{\mathcal{C}} R^{(s^n)}.$$

The co-local elementary subgroup $\mathrm{E}(P, R^{(s^\infty)}) \leq \mathrm{Aut}(P^{(s^\infty)})$ is not a subgroup of $\mathrm{Aut}(P)$ and of $\mathrm{Aut}(P_s)$. But $\mathrm{Aut}(P_s)$ naturally acts on $\mathrm{Aut}(P^{(s^\infty)})$ and normalizes $\mathrm{E}(P, R^{(s^\infty)})$, so everything still works.

Co-local Steinberg groups

From now suppose that the rank of P is at least 4 at every maximal ideal.

Similarly to $\mathrm{E}(P, s^\infty R)$ we can define Steinberg group $\mathrm{St}(P, s^\infty R)$ using a basis of P_s, but in larger category

$$\mathcal{C} = \mathrm{Ex}(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}et))).$$

The external *exact completion* makes $\mathcal{C}$ an *infinitary pretopos*, i.e. there are nice factor-objects of objects by equivalence relations. This allows us to construct groups by generators and relations.

Theorem

The group $\mathrm{St}(P, s^\infty R)$ is perfect. Under the isomorphism $P_s \cong R_s^n$ the group $\mathrm{St}(n, R_s) \rtimes \mathrm{D}(n, R_s)$ canonically acts on $\mathrm{St}(P, s^\infty R)$.

Crossed modules

Recall that in the split case $\mathrm{St}(n, R) \to \mathrm{GL}(n, R)$ is a crossed module in a unique way.

Definition

A *crossed module* is a homomorphism $\delta \colon X \to G$ of groups together with an action $(g, x) \mapsto {}^{g}x$ of G on X such that $\delta({}^{g}x) = g\,\delta(x)\,g^{-1}$ and $xyx^{-1} = {}^{\delta(x)}y$ for $x, y \in X$ and $g \in G$.

Example

Any normal subgroup $N \trianglelefteq G$ is a crossed module. Any perfect central extension $H \twoheadrightarrow G$ is also a crossed module, the action of G on H is unique.

Theorem

The homomorphism $\mathrm{St}(P, s^{\infty} R) \to \mathrm{St}(n, R_s) \rtimes \mathrm{D}(n, R_s)$ *is a crossed module in a unique way.*

Cosheaf property

Suppose that $\mathcal{D}(s) = \bigcup_{i=1}^{N} \mathcal{D}(s_i)$, so $s^\infty R = \sum_i s_i^\infty R$.

Theorem

The abelian group $s^\infty R$ has generators $s_i^\infty R \to s^\infty R$ and relations

$$\begin{array}{ccc} (s_i s_j)^\infty R & \longrightarrow & s_i^\infty R \\ \downarrow & \circlearrowright & \downarrow \\ s_j^\infty R & \longrightarrow & s^\infty R. \end{array}$$

Theorem

The crossed module $\mathrm{St}(P, s^\infty R)$ over $\mathrm{St}(n, R_s) \rtimes \mathrm{D}(n, R_s)$ is generated by $\mathrm{St}(P, s_i^\infty R) \to \mathrm{St}(P, s^\infty R)$ with the only relations

$$\begin{array}{ccc} \mathrm{St}(P, (s_i s_j)^\infty R) & \longrightarrow & \mathrm{St}(P, s_i^\infty R) \\ \downarrow & \circlearrowright & \downarrow \\ \mathrm{St}(P, s_j^\infty R) & \longrightarrow & \mathrm{St}(P, s^\infty R). \end{array}$$

Centrality of K_2

Theorem

There is a unique action $\mathrm{Aut}(P_s)$ *on* $\mathsf{St}(P, s^\infty R)$ *making* $\mathsf{St}(P, s^\infty R) \to \mathrm{Aut}(P_s)$ *a crossed module.*

It is easy to construct actions of individual elements $g \in \mathrm{Aut}(P_s)$, but $\mathcal{C}$ is not Cartesian closed and this does not give an action of $\mathrm{Aut}(P_s)$ as a whole.

The action of $\mathrm{Aut}(P_s)$ can be constructed using generic element method, for this we replace $\mathcal{C} = \mathrm{Ex}(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}et)))$ by

$$\mathcal{C} = \mathrm{Ex}(\mathrm{Ind}(\mathrm{Pro}(\mathcal{C}at(\mathcal{A}lg_R^{\mathrm{fp}}, \mathcal{S}et))))$$

Here $\mathcal{A}lg_R^{\mathrm{fp}}$ is the category of finitely presented unital commutative algebras over R. The finite presentability makes this category small to avoid set-theoretic issues.

Locally isotropic Steinberg groups

Choose $s_1, \ldots, s_N \in R$ such that R_{s_i} are free and $\mathrm{Spec}(R) = \bigcup_{i=1}^N \mathcal{D}(s_i)$.

Definition

The Steinberg group $\mathrm{St}(P, R)$ is the crossed module over $\mathrm{Aut}(P)$ generated by $\mathrm{St}(P, s_i^\infty R) \to \mathrm{St}(P, R)$ with the relations

$$\begin{array}{ccc} \mathrm{St}(P, (s_i s_j)^\infty R) & \longrightarrow & \mathrm{St}(P, s_i^\infty R) \\ \downarrow & \circlearrowleft & \downarrow \\ \mathrm{St}(P, s_j^\infty R) & \longrightarrow & \mathrm{St}(P, R). \end{array}$$

The cosheaf $\mathcal{D}(s) \mapsto \mathrm{St}(P, s^\infty R)$ can be defined on all principal open sets.

Theorem (V.'25)

The Steinberg group $\mathrm{St}(P, R)$ is independent of choices. It perfect, a central extension of $\mathrm{E}(P, R)$, and a crossed module over $\mathrm{Aut}(P)$ in a unique way. If $P \cong R^n$, then $\mathrm{St}(P, R) \cong \mathrm{St}(n, R)$.

Schur multipliers of Steinberg groups

Suppose that the rank n of P is constant.

Theorem (V.'25)

If $n \geq 5$, then $\mathrm{St}(P, R)$ is the universal central extension of $\mathrm{E}(P, R)$. If $n = 4$, then

$$\mathrm{M}(\mathrm{St}(P, R)) \cong R/\langle 2, (r^2 - r)(s^2 - s)\rangle.$$

Theorem (V.'25)

The Steinberg group $\mathrm{St}(P, R) \in \mathcal{C}$ is isomorphic to an object from $\mathrm{Ex}(\mathrm{Ind}(\mathcal{S}et))$.

Proof.

The group $\widehat{\mathrm{St}}(P, R) = \widehat{\mathrm{E}}(P, R)$ lies in $\mathrm{Ex}(\mathrm{Ind}(\mathcal{S}et))$ and the kernel of $\widehat{\mathrm{St}}(P, R) \twoheadrightarrow \mathrm{St}(P, R)$ is just a set. □

Part III. Reductive groups

Chevalley groups

Now let Φ be a reduced irreducible root system of rank ℓ and $\mathrm{G}^{\mathrm{sc}}(\Phi, R)$ be the simply laced Chevalley group of type Φ over R.

Example

$\mathrm{G}^{\mathrm{sc}}(\mathsf{A}_\ell, R) = \mathrm{SL}(\ell + 1, R)$ for $\ell \geq 1$, $\mathrm{G}^{\mathrm{sc}}(\mathsf{C}_\ell, R) = \mathrm{Sp}(2\ell, R)$ for $\ell \geq 1$.

Theorem (Taddei'86, Hazrat–Vavilov'03, Abe'89, Costa–Keller'92–99)

Suppose that $\ell \geq 2$. The elementary subgroup $\mathrm{E}(\Phi, R) \leq \mathrm{G}^{\mathrm{sc}}(\Phi, R)$ is normal.

It is perfect unless $\Phi \in \{\mathsf{B}_2, \mathsf{G}_2\}$ and R has residue fields $\mathbb{F}_2$. The factor-group $\mathrm{G}^{\mathrm{sc}}(\Phi, R)/\mathrm{E}(\Phi, R)$ is solvable if R is finite dimensional.

There is the classification of subgroups of $\mathrm{G}^{\mathrm{sc}}(\Phi, R)$ normalized by $\mathrm{E}(\Phi, R)$.

Steinberg Chevalley groups

Theorem (Lavrenov–Sinchuk–V., '15–'24)

If $\ell \geq 3$, then the Steinberg Chevalley group $\mathrm{St}(\Phi, R)$ is a perfect central extension of $\mathrm{E}(\Phi, R)$.

Theorem (van der Kallen–Stein'77)

Suppose that $\ell \geq 3$. The Steinberg Chevalley group $\mathsf{St}(\Phi, R) \to \mathsf{E}(\Phi, R)$ is the universal central extension unless $\Phi \in \{\mathsf{A}_3, \mathsf{B}_3, \mathsf{C}_3, \mathsf{D}_4, \mathsf{F}_4\}$ and R has residue fields $\mathbb{F}_2$ or $\mathbb{F}_3$. In all cases there is explicit formula for $\mathrm{M}(\mathsf{St}(\Phi, R))$.

Theorem (Matsumoto'69)

If R is a field, then there is an explicit formula for $\mathrm{Ker}\big(\mathsf{St}(\Phi, R) \to \mathsf{E}(\Phi, R)\big)$. It depends only on R and on whether $\Phi \in \{\mathsf{A}_1, \mathsf{B}_2, \mathsf{C}_{\geq 3}\}$ or not.

Locally isotropic reductive groups

All above constructions generalize to locally isotropic reductive groups. Let G be a reductive group scheme over R, e.g. $\mathcal{A}ut(P)$. For simplicity assume that $G/\mathrm{C}(G)$ is geometrically simple, i.e. it is a twisted form of the Steinberg group scheme $\mathrm{G}^{\mathrm{ad}}(\Phi, -)$ for an irreducible root system Φ in étale topology.

If R is local, then G has a maximal split torus $\mathbb{G}_{\mathrm{m}}^k \cong T \leq G$, unique up to conjugation by elements of $G(R)$. Let the *relative root system* $\Psi \leq \mathrm{Hom}(T, \mathbb{G}_{\mathrm{m}}) \cong \mathbb{Z}^k$ be the set of non-zero weights of the action of T on $\mathrm{Lie}(G)$. The set Ψ is empty or an irreducible root system, possibly BC_ℓ.

The *local isotropic rank* of G is the smallest rank of these Ψ calculated by $G_{\mathfrak{m}}$, where $\mathfrak{m} \trianglelefteq R$ runs over all maximal ideals.

Root subgroups

If R is local and $T \leq G$ is a maximal split torus, then there is the scheme centralizer $L = \mathrm{C}_G(T)$ and *root subgroups* $U_\alpha \leq G$ for all $\alpha \in \Phi$.

- The group subscheme L normalizes all U_α.
- If $\Psi = \mathsf{BC}_\ell$ and α is ultarshort, then U_α is two-step nilpotent with a nilpotent filtration $1 \trianglelefteq U_{2\alpha} \trianglelefteq U_\alpha$. Otherwise U_α is abelian.
- The *Chevalley commutator formula*: $[U_\alpha, U_\beta] \leq \prod_{\substack{i\alpha+j\beta\in\Psi \\ i,j>0}} U_{i\alpha+j\beta}$.
- The subgroup $U^+ = \langle U_\alpha \mid \alpha > 0 \rangle \leq G$ is closed and the product morphism $\prod_{0<\alpha\in\Psi\setminus 2\Psi} U_\alpha \to U^+$ is an isomorphism. Similarly for U^-.
- The subgroups $P^\pm = U^\pm \rtimes L$ are closed. The product morphism $U^- \times L \to U^+ \to G$ is an embedding of an open subscheme.

If R is arbitrary, this holds in a small neighborhood of any maximal ideal $\mathfrak{m} \trianglelefteq R$.

Groups of type A_ℓ

Example

Let $G = \mathcal{A}ut(P)$ for a faithful finite projective R-module P. Then $G_\mathfrak{m} \cong \mathcal{GL}(n, R_\mathfrak{m})$, where n is the rank of $P_\mathfrak{m}$. Its root system is

$$\mathsf{A}_{n-1} = \{\mathrm{e}_i - \mathrm{e}_j \mid 1 \leq i \neq j \leq n\} \subseteq \mathbb{R}^n$$

and the local isotropic rank is the minimum of $n-1$ taken over all maximal ideals.

Inside $G_\mathfrak{m}$ the group subscheme L consists of diagonal matrices and $U_{\mathrm{e}_i-\mathrm{e}_j} = \mathrm{Im}(t_{ij})$.

In general if R local and G has relative root system A_ℓ with $\ell \geq 1$, then G is isomorphic to $\mathcal{GL}(\ell+1, A)$ (modulo centers) for an Azumaya algebra A over R, such as $\mathbb{H}$ over $\mathbb{R}$.

Globally isotropic groups

A locally isotropic group G is called *globally isotropic* if it has a proper parabolic subgroup $P < G$. In this case the elementary subgroup is $\mathrm{E}_G(R) = \langle U_P(R), U_P^-(R) \rangle$.

Theorem (Petrov–Stavrova'08, Luzgarev–Stavrova'12, Stavrova–Stepanov'18)

Suppose that G is globally isotropic and of local isotropic rank ≥ 2. Then $\mathrm{E}_G(R)$ is independent of P, it is normal. If R has no residue fields $\mathbb{F}_2$ or the absolute root system Φ is neither B_2 nor G_2, then $\mathrm{E}_G(R)$ is perfect.

If all structure constants are invertible, then there is the classification of subgroups of $G(R)$ normalized by $\mathrm{E}_G(R)$.

Elementary subgroups of reductive groups

Theorem (V.'24)

If the local isotropic rank of G is at least 2, then $G(R)$ has a canonical elementary subgroup $\mathrm{E}_G(R) \trianglelefteq G(R)$. This subgroup is normal and it has a canonical ind-structure given by a generating morphism $\mathbb{A}^N \to G$.

If $P \leq G$ is a proper parabolic subgroup, then $\mathrm{E}_G(R) = \langle U_P(R), U_P^-(R) \rangle$. If $G = \mathrm{G}(\Phi, -)$ is a Chevalley group scheme, then $\mathrm{E}_G(R) = \mathrm{E}(\Phi, R)$.

Suppose that for every $\mathfrak{m} \trianglelefteq R$ with the residue field $\mathbb{F}_2$ the group scheme $(G/\mathrm{C}(G))_{R/\mathfrak{m}}$ is neither $G^{\mathrm{ad}}(\mathrm{B}_2, -)$ nor $G^{\mathrm{ad}}(\mathrm{G}_2, -)$. Then $\mathrm{E}_G(R)$ is perfect.

Example

The commutator subgroups of $\mathrm{B}_2(2) = \mathrm{E}^{\mathrm{ad}}(\mathrm{B}_2, \mathbb{F}_2)$ and $\mathrm{G}_2(2) = \mathrm{E}^{\mathrm{ad}}(\mathrm{G}_2, \mathbb{F}_2)$ have index 2.

Steinberg groups of reductive groups

Theorem (V.'25)

If the local isotropic rank of G is at least 3, then there is a canonical Steinberg group $\mathrm{St}_G(R)$. It is perfect and a crossed module over $G(R)$.

If $G = \mathrm{G}(\Phi, R)$ is a Chevalley group scheme, then $\mathrm{St}_G(R) = \mathrm{St}(\Phi, R)$.

Theorem (V.'25)

The Schur multiplier $\mathrm{M}(\mathrm{St}_G(R))$ is calculated in terms of R and G, it is trivial if the local isotropic rank of G is at least 5.

The Steinberg group $\mathrm{St}_G(R)$ is an object in $\mathrm{Ex}(\mathrm{Ind}(\mathcal{S}et))$, so it can be considered as an abstract group using the evaluation functor $\mathrm{Ex}(\mathrm{Ind}(\mathcal{S}et)) \to \mathcal{S}et$.

Finite groups of Lie type

Recall that a *finite group of Lie type* is $\Phi(q) = \mathsf{E}^{\mathrm{ad}}(\Phi, \mathbb{F}_q)$, where

$$\begin{aligned} \Phi \in \{&\mathsf{A}_{\geq 1}, \mathsf{B}_{\geq 2}, \mathsf{C}_{\geq 3}, \mathsf{D}_{\geq 4}, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8, \mathsf{F}_4, \mathsf{G}_2\} && \text{(Chevalley groups)} \\ \sqcup \{&{}^2\mathsf{A}_{\geq 2}, {}^2\mathsf{D}_{\geq 4}, {}^2\mathsf{E}_6, {}^3\mathsf{D}_4\} && \text{(Steinberg variations)} \\ \sqcup \{&{}^2\mathsf{B}_2, {}^2\mathsf{F}_4, {}^2\mathsf{G}_2\}, && \text{(Suzuki and Ree groups)} \end{aligned}$$

in the second row q is a square or a cube depending on the upper index, and in the last row $q = 2^{2k+1}$, $q = 2^{2k+1}$, $q = 3^{2k+1}$ respectively.

Chevalley groups and Steinberg variations are elementary subgroups of the point groups of adjoint reductive group schemes over $\mathbb{F}_q$, $\mathbb{F}_{\sqrt{q}}$, or $\mathbb{F}_{\sqrt[3]{q}}$ respectively. The relative root system Ψ is

Φ	${}^2\mathsf{A}_{2k}$	${}^2\mathsf{A}_{2k-1}$	${}^2\mathsf{D}_{k+1}$	${}^2\mathsf{E}_6$	${}^3\mathsf{D}_4$	${}^2\mathsf{B}_2$	${}^2\mathsf{F}_4$	${}^2\mathsf{G}_2$
Ψ	BC_k	C_k	B_k	F_4	G_2	A_1	I_2^8	A_1
k	≥ 1	≥ 2	≥ 3					

Schur multipliers of finite groups of Lie type

Finite groups of Lie type are simple except

$$A_1(2),\ A_1(3),\ B_2(2),\ G_2(2),\ {}^2A_2(4),\ {}^2B_2(2),\ {}^2F_4(2),\ {}^2G_2(3).$$

Always $\mathrm{G}^{\mathrm{sc}}(\Phi, \mathbb{F}_q) = \mathrm{E}^{\mathrm{sc}}(\Phi, \mathbb{F}_q) = \mathrm{St}(\Phi, \mathbb{F}_q)$. If $\Phi(q)$ is simple, then $\mathrm{St}(\Phi, \mathbb{F}_q) = \widehat{\Phi(q)}$ with the following exceptions.

$\Phi(q)$	$A_1(4)$	$A_1(9)$	$A_2(2)$	$A_2(4)$	$A_3(2)$	$B_3(2)$
$\mathrm{M}(\mathrm{St}(\Phi, \mathbb{F}_q))$	C_2	C_3	C_2	$\mathrm{C}_4 \times \mathrm{C}_4$	C_2	C_2

$\Phi(q)$	$B_3(3)$	$C_3(2)$	$F_4(2)$	$G_2(3)$	$G_2(4)$	$D_4(2)$
$\mathrm{M}(\mathrm{St}(\Phi, \mathbb{F}_q))$	C_3	C_2	C_2	C_3	C_2	$\mathrm{C}_2 \times \mathrm{C}_2$

$\Phi(q)$	${}^2A_3(4)$	${}^2A_3(9)$	${}^2A_5(4)$	${}^2E_6(4)$	${}^2B_2(8)$
$\mathrm{M}(\mathrm{St}(\Phi, \mathbb{F}_q))$	C_2	$\mathrm{C}_3 \times \mathrm{C}_3$	$\mathrm{C}_2 \times \mathrm{C}_2$	$\mathrm{C}_2 \times \mathrm{C}_2$	$\mathrm{C}_2 \times \mathrm{C}_2$

Schur multipliers of reductive Steinberg groups

The Schur multiplier of $\mathrm{St}_G(R)$ depends on the root system Φ of the split form of G, i.e. the *absolute root system*.

Theorem (V.'25)

If the local isotropic rank of G is at least 3, then the Schur multiplier $M = \mathrm{M}(\mathrm{St}_G(R))$ is trivial with the following exceptions.

Φ	A_3	A_5	B_3	C_3	D_4	F_4	E_6
M	$R_{2\varepsilon}$	$R_{2\mathrm{a}} \times R_{2\mathrm{a}}$	$R_3 \times R_{2\varepsilon}$	R_2	$R_{2\mathrm{s}} \times R_{2\mathrm{s}\varepsilon}$	R_2	$R_{2\mathrm{a}} \times R_{2\mathrm{a}}$

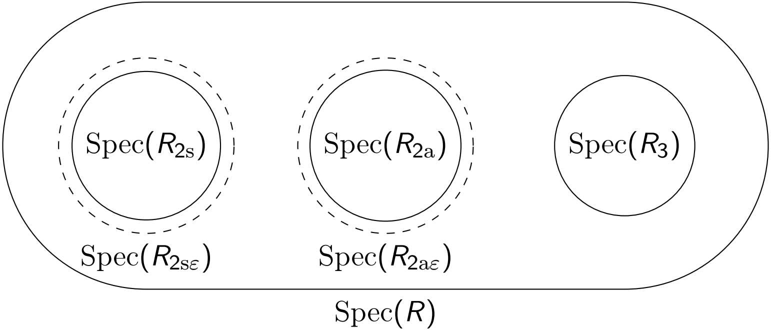
Here

$$R_2 = R/\langle 2, p^2 - p\rangle, \;\; R_{2\varepsilon} = R/\langle 2, (p^2 - p)(q^2 - q)\rangle, \;\; R_3 = R/\langle 3, p^3 - p\rangle.$$

There is a decomposition $R_2 = R_{2\mathrm{s}} \times R_{2\mathrm{a}}$ such that $G/\mathrm{C}(G)$ splits over $R/\mathfrak{m}$ for $\mathfrak{m} \in \mathrm{Spec}(R_{2\mathrm{s}})$ and does not split for $\mathfrak{m} \in \mathrm{Spec}(R_{2\mathrm{a}})$. Finally, $R_{2\varepsilon} = I \rtimes R_2$ for $I = \langle p^2 - p\rangle \trianglelefteq R_{2\varepsilon}$, $I^2 = 0$, and

$$R_{2\mathrm{s}\varepsilon} = I \cdot R_{2\mathrm{s}} \rtimes R_{2\mathrm{s}}, \quad R_{2\mathrm{a}\varepsilon} = I \cdot R_{2\mathrm{a}} \rtimes R_{2\mathrm{a}}.$$

Schur multipliers inside $\mathrm{Spec}(R)$



Example

If $R = \mathbb{Z}[a, (1-4a)^{-1}]$ and $G = \mathcal{SO}(R^7, u^2 + vw + xy + z^2 + zw + aw^2)$, then $R_{2\mathrm{s}} \cong R_{2\mathrm{a}} \cong \mathbb{F}_2$, $R_3 \cong \mathbb{F}_3 \times \mathbb{F}_3$, $R_{2\mathrm{s}\varepsilon} \cong R_{2\mathrm{a}\varepsilon} \cong \mathbb{F}_2[\varepsilon]/(\varepsilon^2)$.