Abstract isomorphisms of isotropic root graded groups over rings

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Part 1. Examples

Projective linear groups over Azumaya algebras

Definition

An associative algebra A over commutative ring R is called *Azumaya algebra* if

$$A \otimes_R S \simeq M_{n\times n}(S)$$

as an S-algebras, where S ia a faithfully flat extension of R.

Corollary of the main theorem

Let A_1 and A_2 be two algebras such that each is an Azumaya algebra over its center. Suppose that 2 is invertible in the center of A_2. Let $n \geqslant 2$ and $m \geqslant 2$ be integers.

Then any isomorphism between the groups $\mathrm{PGL}_n(A_1)$ and $\mathrm{PGL}_m(A_2)$ arises from an involution-preserving isomorphism of the matrix rings $M_{n\times n}(A_1 \times A_1^{\mathrm{op}})$ and $M_{m\times m}(A_2 \times A_2^{\mathrm{op}})$.

Projective anti-hermitian unitary groups

Let A_1 be the Azumaya algebra over R_1 with an R_1-linear orthogonal involution; V_1 – projective module of constant rank over A; h_1 be a non-degenerate anti-hermitian form on V_1 of Witt index r_1.
Similarly for A_2, R_2, V_2, h_2, r_2.
Assume that:

- $r_1 \geqslant 2$ and $r_2 \geqslant 2$;
- $2 \in R_1^*$ and $2 \in R_2^*$;
- [Assumption (d)]: A_1 contains an invertible anti-hermitian element.

Corollary of the main theorem

In the setting above any isomorphism between the groups $\mathrm{PU}(V_1, h_1)$ and $\mathrm{PU}(V_2, h_2)$ arises from an isomorphism of the rings $\mathrm{End}_{A_1}(V_1)$ and $\mathrm{End}_{A_2}(V_2)$ that turns transpose w.r.t h_1 into transpose w.r.t h_2.

Projective orthogonal groups

Let V_k be a projective module of constant rank over R_k; q_k be a semi-regular quadratic form on V_k of Witt index r_k ($k = 1{,}2$).
Assume that:

- $r_1 \geqslant 2$ and $r_2 \geqslant 2$;
- $2 \in R_1^*$ and $2 \in R_2^*$;
- [Assumption (d)]: If $r_1 = 2$, then $V_1 = \mathcal{H} \oplus \mathcal{H} \oplus W$ (the sum is orthogonal $\mathcal{H}$ — hyperbolic plane), such that there are orthogonal elements $v, w \in W$ with $q_1(v)$,$q_1(w) \in R_1^*$.

Corollary of the main theorem

In the setting above for any isomorphism between the groups $\mathrm{PSO}(V_1, q_1)$ and $\mathrm{PSO}(V_2, q_2)$ there is a ring isomorphism $\varphi\colon R_1 \xrightarrow{\sim} R_2$ and the projective module P of rank 1 over R_2 with a non-degenerate bilinear form $(\cdot, \cdot)$ such that the isomorphism in question arises from an isomorphism of quadratic R_2-modules $(V_1^{\varphi}, \varphi(q_1))$ and $(V_2 \otimes_{R_2} P, q_2')$, where
$q_2'(\sum_i v_i \otimes p_i) = \sum_i q_2(v_i)(p_i, p_i) + \sum_{i<j} b_2(v_i, v_j)(p_i, p_j)$;
$b_2(v_i, v_j) = q_2(v_i + v_j) - q_2(v_i) - q_2(v_j)$.

Part 2. Main Theorem

Main Theorem: Setting

Definition

The group scheme G over the ring R is said to be absolutely simple adjoint with a common root datum of the geometric fibers if there is an fppf-extension S of the ring R such that G_S is isomorphic to the adjoint Chevalley–Demazure scheme $G_{\mathrm{ad}}(\widetilde{\Phi}, -)_S$ over S.
The root system $\widetilde{\Phi}$ here is called the *absolute root system* of G and is assumed to be irreducible.

Let G_1 and G_2 be absolutely simple adjoint group schemes over rings R_1 and R_2, each with a common root datum of the geometric fibers. Let $\widetilde{\Phi_1}$ and $\widetilde{\Phi_2}$ be the corresponding absolute root systems.
Suppose that we have intermediate subgroups $E_1(R_1) \leqslant H_1 \leqslant G_1(R_1)$ and $E_2(R_2) \leqslant H_2 \leqslant G_2(R_2)$. Let

$$\theta\colon H_1 \xrightarrow{\sim} H_2$$

be the isomorphism of abstract groups.

Main Theorem: Assumptions

- G_2 has isotropic rank at least 2;
- G_1 admits an isotropic pinning with the following properties:

a) its root system Φ has rank at least 2;

b) the corresponding map $u\colon \widetilde{\Phi_1} \cup \{0\} \to \Phi \cup \{0\}$ comes from one of the Tits indexes;

c) it has square formula;

d)if Φ is of type C or BC (including C_2) and the map u is not a bijection, then for every pair of orthogonal short roots $\beta,\beta' \in \Phi$ with their sum being a long root the corresponding Weyl elements $w_\beta = a_\beta b_\beta c_\beta$ and $w_{\beta'} = a_{\beta'} b_{\beta'} c_{\beta'}$ (here a_β, $c_\beta \in (G_1)_\beta$ and $b_\beta \in (G_1)_{-\beta}$; and similarly for $a_{\beta'}$, $b_{\beta'}$, $c_{\beta'}$) can be chosen so that b_β commutes with $b_{\beta'}$;

- if $\widetilde{\Phi_1}$ is doubly laced, then $2 \in R_1^*$; if $\widetilde{\Phi_1} = G_2$, then $6 \in R_1^*$;
- $2 \in R_2^*$ (regardless of $\widetilde{\Phi_2}$); if $\widetilde{\Phi_2} = G_2$, then $6 \in R_2^*$.

Main Theorem: Conclusion

Then

1. If $\widetilde{\Phi_1}$ is not isomorphic to $\widetilde{\Phi_2}$, then $\widetilde{\Phi_1} = A_3$, $\widetilde{\Phi_2} = B_2$, $R_1/\mathfrak{M} \simeq \mathbb{F}_2$ for all maximal ideals $\mathfrak{M} \trianglelefteq R_1$ and $R_2/\mathfrak{M} \simeq \mathbb{F}_3$ for all maximal ideals $\mathfrak{M} \trianglelefteq R_2$.
2. If $\widetilde{\Phi_1} = \widetilde{\Phi_2}$, then there exists an isomorphism of rings $\varphi\colon R_1 \xrightarrow{\sim} R_2$ and an R_2-group-scheme isomorphism $\Theta\colon {}^{\varphi}G_1 \xrightarrow{\sim} G_2$ such that $\theta = (\Theta_{R_2} \circ \varphi_*)|_{H_1}$, where $\varphi_*\colon G_1(R_1) \xrightarrow{\sim} {}^{\varphi}G_1(R_2)$ is the isomorphism induced by φ.

Isotropic rank

Definition

We say that the absolutely simple adjoint group scheme G with a common root datum of the geometric fibers has isotropic rank at least l if it contains a closed subgroup-subscheme isomorphic to $\mathbb{G}_m^l$.

- If G has isotropic rank $\geqslant 1$; and locally on the base has isotropic rank $\geqslant 2$, then $E(R)$ is well defined and normal (result by Petrov and Stavrova).
- One can define a root subgroups G_α, $\alpha \in \Phi$. Φ is called a *relative root system*. $\underline{\text{But } \Phi \text{ does not have to be a root system}}$.
- In the extension, where G splits, we have $G_\alpha = \prod_{\gamma \in u^{-1}(\alpha)} G_\gamma$, $u\colon \widetilde{\Phi} \cup \{0\} \to \Phi \cup \{0\}$.
- The map u arises from projecting the ambient vector space of $\widetilde{\Phi}$ onto the quotient by action of $\Gamma \leqslant \mathrm{Aut}(D)$ and by span of $D \setminus J$, where D is the Dynkin diagram of $\widetilde{\Phi}$.

Pinning

Definition (SGA)

A pinning of G consists of

- a maximal torus $T \leqslant G$ with a chosen isomorphism $T \simeq \mathbb{G}_m^l$;
- a root datum $(\mathbb{Z}^l, \Phi, (\mathbb{Z}^l)^\vee, \Phi^\vee)$ such that Φ and $\Phi^\vee$ are the sets of roots and coroots of G with respect to T;
- a basis $\Delta \leqslant \Phi$;
- generators $x_\alpha \in \mathfrak{g}_\alpha$ in the root spaces $\mathfrak{g}_\alpha \leqslant \mathrm{Lie}(G)$ for $\alpha \in \Delta$, so that all root spaces are free R-modules of rank 1.

If G has a pinning, then it is isomorphic to $G(\widetilde{\Phi}, -)_R$ already over R and the root system Φ in the corresponding root datum coincides with the absolute root system $\widetilde{\Phi}$.

Isotropic pre-pinning

Definition (Voronetsky)

An isotropic pre-pinning of G consists of

- a split torus $T \leqslant G$ with a chosen isomorphism $T \simeq \mathbb{G}_m^l$;
- a root system $\Phi \subseteq \mathbb{Z}^l$ (with respect to some inner product on $\mathbb{R}^l$)

together with a chosen base $\Delta \leqslant \Phi$ such that Φ is the set of non-zero weights of $\mathfrak{g}$ with respect to T;

- Φ is assumed to be irreducible, crystallographic, but not necessarily reduced (i.e., it may be of type BC).
- {Pinnings}$\subset$ {Isotropic pinnings}$\subset$ {Isotropic pre-pinnings};
- "G has isotropic pre-pinning with $\mathrm{rk}\,\Phi \geqslant r$" is stronger than "$G$ has isotropic rank $\geqslant r$";
- Φ is the relative root system of G;

Isotropic pinning

For a root $\alpha \in \Phi$, an element $w_\alpha \in G_\alpha(R)G_{-\alpha}(R)G_\alpha(R)$ is called a *Weyl element* if ${}^{w_\alpha}G_\beta(S) = G_{s_\alpha(\beta)}(S)$ for every root $\beta \in \Phi$ and any R-algebra S.

Definition (Voronetsky)

An isotropic pre-pinning (T, Φ) is called an *isotropic pinning* if Weyl elements w_α exist for every $\alpha \in \Phi$.

Example

$P = P_1 \oplus \ldots \oplus P_{n+1}$, P_i — projective modules of constant rank over R.
$G(S) = \mathrm{PAut}_S(P \otimes_R S)$.

- G always has isotropic rank $\geqslant n$.
- If all $\mathrm{rk}\, P_i$ are equal, them G has isotropic pre-pinning with $\Phi = A_n$.
- If all P_i are isomorphic, then G has isotropic pinning with $\Phi = A_n$.
- If all $P_i = R$, then G has a pinning ($G \simeq \mathrm{PGL}_{n+1}$).

Tits indices

Recall: $u\colon \widetilde{\Phi} \cup \{0\} \to \Phi \cup \{0\}$ arises from $J \subseteq D$ and $\Gamma \subseteq \mathrm{Aut}(D)$.

Definition

The triple $(\widetilde{\Phi}, \Gamma, J)$ is called a Tits index, if it can be constructed by a reductive group scheme over a field using a torus that is maximal among split tori.

- There is a list of Tits indexes.
- An absolutely simple group scheme over a field K is uniquely defined by: 1) Its root datum over $\overline{K}$; 2) Its Tits index; 3) Action of the Galois group on the Dynkin diagram $\mathrm{Gal}(\overline{K}/K) \twoheadrightarrow \Gamma$; 4) Its anisotropic Kernel (centralizer of split torus).
- The same is true for groups over local rings, with the same list of Tits indexes (result by Petrov and Stavrova).

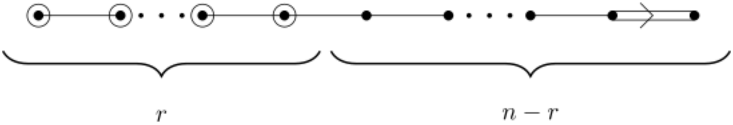
Some Tits indices

- ${}^1A_{n,r}^{(d)}$, $d(r+1) = n+1$. Correspond to $\mathrm{PGL}_r(A)$, A— Azumaya algebra of rank d^2. $\widetilde{\Phi} = A_n$, $\Phi = A_r$

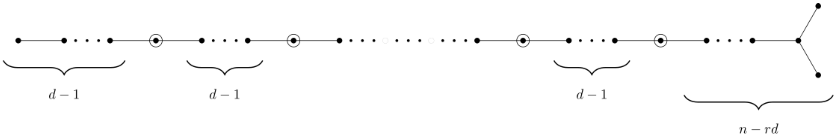
- $C_{n,r}^{(d)}$. Correspond to $\mathrm{PU}(V,h)$, V — projective module of constant rank over Azumaya algebra over R with R-linear orthogonal involution; h — anti-hermitian form on V of Witt index r. $\widetilde{\Phi} = C_n$, $\Phi = C_r$ if $n = rd$. $\Phi = BC_r$ otherwise.

Some Tits indices

- $B_{n,r}$, $d(r+1) = n+1$. Correspond to $\mathrm{PSO}(V,q)$, $\mathrm{rk}\, V = 2n+1$, Witt index of q is $\geqslant r$. $\widetilde{\Phi} = B_n$, $\Phi = B_r$.



- ${}^1D_{n,r}^{(d)}$, $\mathrm{PU}(V,h)$, V — projective module of constant rank over Azumaya algebra over R with R-linear orthogonal involution; h — hermitian form on V of Witt index r. $\widetilde{\Phi} = D_n$, Φ could be C_r, B_r or BC_r.



Square formula

- If $\alpha \in \Phi$ is not ultrashort, then $G_\alpha(S)$ is an abelian group with a natural structure of S-module (for any commutative R-algebra S).
- If α is ultrashort, we have $G_{2\alpha}(S) \leqslant G_\alpha(S)$ and there is a natural action of the multiplicative monoid $S^\bullet$ on $G_\alpha(S)$, which induces an S-module structure on $G_\alpha(S)/G_{2\alpha}(S)$
- There are elements $h_\alpha \in G(R)$, $\alpha \in \Phi$ such that for any $u \in G_\beta(S)$, $u^{h_\alpha} = u \cdot (-1)^{\langle \beta,\alpha \rangle}$.

Definition

We say that an isotropic pinning (T, Φ) has *square formula*, if the Weyl elements w_α can be chosen so that $w_\alpha^2 = h_\alpha$ for all $\alpha \in \Phi$.

- True for the cases, where either Φ is simply-laced or $\mathrm{rk}\,\Phi \geqslant 3$ (follows from results of Wiedemann)
- Probably always true.

Part 3. Some previous results to compare with

Borel–Tits theorem

Let K_1, K_2 be infinite fields. Let G_1, G_2 be absolutely almost simple isotropic group schemes over K_1 and K_2. Let $G_1^+(K_1) \leqslant H_1 \leqslant G_1(K_1)$ and $G_2^+(K_2) \leqslant H_2 \leqslant G_2(K_2)$. Let $\alpha\colon H_1 \to H_2$ be a surjective homomorphism with $\alpha(G_1^+(K_1)) \neq \{e\}$. Then

1. There is a unique field isomorphism $\varphi\colon K_1 \xrightarrow{\sim} K_2$ and a unique K_2-isogeny $\beta'\colon {}^{\varphi}G_1 \to \mathrm{Ad}\, G_2$ such that $\beta' \circ \varphi^*|_{H_1} = \mathrm{Ad}_{G_2} \circ \alpha$.
2. Unless G_1 and G_2 are forms of orthogonal or half-spin groups of type D_{2n}, if Ad_{G_1} and Ad_{G_2} have the same degree, then there is an isogeny $\beta\colon {}^{\varphi}G_1 \to G_2$ and a homomorphism $\mu\colon H_1 \to \mathrm{Cent}(G_2(K_2))$ such that $\alpha(h) = \mu(h) \cdot \beta(\varphi^*(h))$.
3. The only situations, where β' is non-central or β is not an isomorphism are as follows:
   a) $\mathrm{Char} K_i = 2$, G_1 is of type $\mathrm{B_n}$, resp. $\mathrm{C_n}$, resp. $\mathrm{F_4}$; G_2 is of type $\mathrm{C_n}$, resp. $\mathrm{B_n}$, resp. $\mathrm{F_4}$.
   b) $\mathrm{Char} K_i = 3$, G_1 and G_2 is of type $\mathrm{G_2}$.
   In both cases G_1, G_2 must be split, and K_1, K_2 must be perfect.

Bunina's results on Chevalley groups

- Assume that the Chevalley groups $G(\Phi_1, R_2)$ and $G(\Phi_2, R_2)$ are isomorphic. If Φ_1 resp. Φ_2 is of type $\mathrm{A_2}$,$\mathrm{B_n}$,$\mathrm{C_n}$ or $\mathrm{F_4}$ assume that R_1 resp. R_2 has $1/2$; if Φ_1 resp. Φ_2 is of type $\mathrm{G_2}$ assume that R_1 resp. R_2 has $1/6$. Then $\Phi_1 \simeq \Phi_2$ and $R_1 \simeq R_2$.
- Under the same assumptions about $1/2$ and $1/6$, any automorphism of the Chevalley group $G(\Phi, R)$ is a composition of ring automorphism, scheme automorphism (''inner'' + graph) and a central automorphism. (The proof contains part that does not work for half-spin groups of type D_{2n}).

Part 4. Model theoretic applications

Elementary equivalence

Let L be a first order language. An L-structure is a set with a operations, constants and predicates from L.

Elementary equivalence

Structures A and B in the language L are elementary equivalent, if they satisfy the same first-order sentences.

Elementary embeddings

An embedding $i\colon A \to B$ is called elementary, if it preserves first-order properties of any tuple. (In particular A and B are elementary equivalent.)

Ultrafilters

1. If $\mathcal{F}$ is an ultrafilter, then diagonal embedding $i\colon A \to \prod_{\mathcal{F}} A$ is elementary.
2. (Keisler–Shelah theorem) If A and B are elementary equivalent, then there exists an ultrafilter $\mathcal{F}$ such that $\prod_{\mathcal{F}} A \simeq \prod_{\mathcal{F}} B$.

Elementary equivalence

Corollary of the main theorem

Suppose that G_k, R_k, $(k = 1,2)$ be like in the main theorem. Assume that the groups $G_1(R_1)$ and $G_2(R_2)$ are elementary equivalent. Then
either there are a ring S and elementary embeddings $i_1 \colon R_1 \to S$ and $i_2 \colon R_2 \to S$ such that the base changes of G_1 and G_2 to S via i_1 and i_2 are isomorphic as group schemes over S;
or $\widetilde{\Phi}_1 = A_3$, $\widetilde{\Phi}_2 = B_2$, $R_1/\mathfrak{M} \simeq \mathbb{F}_2$ for all maximal ideals $\mathfrak{M} \trianglelefteq R_1$ and $R_2/\mathfrak{M} \simeq \mathbb{F}_3$ for all maximal ideals $\mathfrak{M} \trianglelefteq R_2$.

Proof

$G_1(\prod_{\mathcal{F}} R_1) \simeq \prod_{\mathcal{F}} G_1(R_1) \simeq \prod_{\mathcal{F}} G_2(R_2) \simeq G_2(\prod_{\mathcal{F}} R_2)$.
Apply the theorem and set $S \simeq \prod_{\mathcal{F}} R_1 \simeq \prod_{\mathcal{F}} R_2$.

Definable sets

Definable sets

Let M be an L-structure. The subset of M^n is called **definable** if it can be defined by a first order formula (possibly with parameters from M). If no parameters are required the subset is called **absolutely definable**.

Example

Centralizer of an element in a group is a definable set. Center of a group is an absolutely definable set.

Imaginaries

Given a definable set X and a definable equivalence relation $E \subseteq X \times X$, the set of E-equivalence classes is called an **imaginary**.

Interpretation

Suppose that L_1, L_2 are two (possibly different) first-order languages, and that, for $i = 1, 2$, M_i is a structure of L_i. An interpretation of M_2 in M_1 consist of:

1. A tuple of parameters $p \in M_1^k$.
2. A code Γ that consist of L_1-first-order formulas that use p as parameters and define
   a) An imaginary $\Gamma(M_1, p)$ over M_1;
   b)All operation/constants/predicates from L_2 on the set $\Gamma(M_1, p)$;
3. A coordinate map $\mu\colon \Gamma(M_1, p) \to M_2$, which is an isomorphism of L_2-structures.

Example

Let G be an affine finitely presented group scheme over R, then $G(R)$ is interpretable in R (via system of polynomial equation).
The proof of the main theorem involves construction of interpretation of R in $G(R)$.

Bi-interpretations

Homotopies

A homotopy between two interpretations (Γ, p, μ) and (Γ', p', μ') of M_2 in M_1 is an L_1-first-order formula θ (possibly with parameters) that defines the graph of the map $\mu' \circ \mu^{-1}$.

Bi-interpretation ('strong bi-interpretation" in terminology of Daniyarova and Myasnikov)

The structures M_1 and M_2 are bi-interpretable if there are interpretations (Γ, p, μ) of M_1 in M_2 and (Δ, q, ν) of M_2 in M_1, such that both compositions of these interpretations are homotopic to identity.
M_1 and M_2 are absolutelly bi-interpretable if neither interpretations nor homotopies use parameters.

If M_1 and M_2 are absolutely bi-interpretable, then $\mathrm{Aut}(M_1) \simeq \mathrm{Aut}(M_2)$. Therefore, absolute bi-interpretations are rare.

Regular bi-interpretation

A Regular bi-interpretation consist of:

1. L_1-first order k-ary formula φ and L_2-first order l-ary formula ψ;
2. interpretations codes Γ and Δ with k resp. l parameters;
3. homotopy formulas θ_1 and θ_2;

such that
a)φ resp. ψ define an non-empty set $\varphi(M_1) \subseteq M_1^k$ resp. $\psi(M_2) \subseteq M_2^l$;
b) for any $p \in \varphi(M_1)$, $q \in \psi(M_2)$ there are coordinate maps μ and ν such that

- (Γ, p, μ) is an interpretation of M_2 in M_1, and (Δ, q, ν) is an interpretation of M_1 in M_2;
- for any representative $\widetilde{q}$ of $\mu^{-1}(q)$, the formula θ_1 with parameters $(p, \widetilde{q})$ defines a homotopy between $(\Gamma, p, \mu) \circ (\Delta, q, \nu)$ and identity.
- for any representative $\widetilde{p}$ of $\nu^{-1}(p)$, the formula θ_2 with parameters $(\widetilde{p}, q)$ defines a homotopy between $(\Delta, q, \nu) \circ (\Gamma, p, \mu)$ and identity.

Regular bi-interpretability of Chevalley groups

Theorem (Bunina–Gvozdevsky)

Under the same assumptions as for isomorphism theorem, the adjoint Chevalley group $G_{\mathrm{ad}}(\Phi, R)$ is regularly bi-interpretable with R.

Remark. The proof of regularity uses the isomorphism theorem.

Corollary

Under the same assumptions. For any group G that is elementary equivalent to $G_{\mathrm{ad}}(\Phi, R)$, we have $G \simeq G(\Phi, S)$ with S elementary equivalent to R.

Plans for the future paper

To be proven

Let R and $G(-)$ be as in the main theorem. Suppose that G as a group scheme has a form over a subring of R, which is either $\mathbb{Z}$, or quotient of $\mathbb{Z}$ of localization of $\mathbb{Z}$ (by a multiplicative system). Then $G(R)$ is regularly bi-interpretable with R.

To be proven

Let R_1 and $G_1(-)$ be as in the main theorem. Let G be a group elementary equivalent to $G_1(R_1)$. Then there are rings R_2 and S, elementary embeddings $i_1\colon R_1 \to S$ and $i_2\colon R_2 \to S$, and a group scheme G_2 over R_2 such that $G \simeq G_2(R_2)$ and the base changes of G_1 and G_2 to S via i_1 and i_2 are isomorphic as group schemes over S.

Thank you for your attention.