Jacobi inversion problem: classical and contemporary aspects

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Abel transform sends divisors on a Riemann surface Σ into Abelian variety called Jacobian $Jac(\Sigma)$ of the surface. The inverse map is given by Riemann vanishing theorem which claims that the preimage of a generic point of the Jacobian can be found as the divisor of zeroes of a certain auxiliary function F on the Riemann surface, where F is constructed with help of the Riemann theta function of the surface. Together, Abel map and the Riemann theorem establish a birational equivalence between $Jac(\Sigma)$ and $\mathrm{Sym}^g\Sigma$ where $g = genus(\Sigma)$.

The problem of reversion of the Abel map is named after Jacobi. Jacobi's motivation can be judged with some degree of probability from his "Lectures on Dynamics". In the "Lectures" he expresses the idea that it is quite stupid to try to find out a change of variables solving a given differential equation. It is more smart to start with some simple dynamics, complicate it with help of some change of variables, and look for a problem which can be described by means of the obtained dynamics. Following that idea he takes uniform rectilinear motion and by means of a certain sophisticated transformation turns it into dynamics $\zeta = \zeta(t)$ satisfying the equation which reads as $A(\zeta) = c(t)$ in our contemporary language, where A is the Abel transform, ζ is a degree g divisor, $c(t) \in Jac(\Sigma)$ is a curve known in a sense it is expressed in terms of the original linear dynamics. The conclusion should be as follows: by reversion of the Abel map A we can find out the dynamics of divisors corresponding to our original linear dynamics.

It turned out to be that for a majority of known completely integrable systems, from classical systems to contemporary soliton equations, the Abel transform linearizes solutions, and the Jacobi inversion gives the dynamical divisor of the system. We will show it with the example of Hitchin systems.

The Riemann theorem reduces solution of the Jacobi problem to a trancendental equation containing theta functions. It is known (Buchstaber–Enolski–Leykin, 2012) that for hyperelliptic curves the problem is algebraic. However, in 1984 Dubrovin proposed an approach (again going back to Riemann) enabling to calculate some symmetric functions of the divisor of zeroes without resolving the equation itself. In particular, it enabled him to derive the celebrated Its-Matveev theta functional formula for solutions of the Korteveg-de Vries equation. Its and Matveev obtained it a bit earlier using similar ideas. In general, the symmetric functions computable by means of Dubrovin's approach do not give full invariants of the trajectory, however calculation of them reduces the Jacobi inversion problem to an algebraic one.

A wide class of finite-dimensional integrable systems is characterized by their spectral curves. These are finite genus algebraic curves over $\mathbb{C}$. It is remarkable that for those systems, the Separation of Variables method leads to an Abel-type map which transforms trajectories to the straight lines on some Abelian variety related to the spectral curve. The type of the map, and of the variety depends on the symmetry group of the integrable system. In physics, the symmetry group is usually referred to as the gauge group. If the gauge group is equal to $GL(n)$ then the above map is the Abel map itself. In the case of simple gauge groups the spectral curve of the system possesses a holomorphic involution coming from the Cartan involution on the group. In this case, the above mentioned map is the Abel–Prym map, and the corresponding Abelian variety is a certain finite unramified covering of the Prymian of the spectral curve.

Consideration of the case of a spectral curve with an involution was pioneered by Novikov and Veselov (1984), with relation to Schrödinger equation, and recently continued by the author, with relation to Hitchin systems. The Riemann vanishing theorem still holds in this case, but it is even less effective than in the classical case, becouse the divisor of zeroes is subject to an additional relation. The author managed to show that in presence of a second involution commuting with the first one the degree of efficiency is the same as in the classical case. It enabled us to integrate the Hitchin systems with gauge groups $SL(2)$ and $SO(4)$. It also implies birational equivalence between the Prymian of the first involution and a certain symmetric power of the quotient of the curve by the second involution.

Search for real solutions is a one more problem of the theory of integrable systems. This direction is related with works by Cherednik, Dubrovin and Natanzon, Veselov and Novikov. In the context of Jacobi inversion it is a problem of realness conditions for trajectories on the Abelian variety in question, which in turn reduces to study the symmenries of theta functions of real curves. The known results are due to Dubrovin and Natanzon for the Riemann theta function, and due to the author for the Prym theta function.

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