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Model Theory of Algebraic Groups. Where to go?

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Based on recent works by E.Bunina(Bar Ilan), E.Voronetsky (SpB), P.Gvozdevsky(Bar Ilan)

Definition

Given an algebra H, its *elementary theory* $Th(H)$ is the set of all sentences (closed formulas) valid on H.

Definition

Two algebras H_1 and H_2 are said to be *elementarily equivalent* if their elementary theories coincide.

Problem

Given an algebra H, describe the isomorphism class of H.

Problem

Given an algebra H, describe the elementary class of H.

Notations $A \cong B$, $A \equiv B$.

Elementary equivalence of linear groups

Let us make emphasis on elementary equivalence of linear groups. Our goal is to discus these problems from the perspectives of the class of affine algebraic groups. So, let H be an affine algebraic group.

WHERE to GO???

Problem

What are the linear algebraic groups which admit a kind of reasonable description of their model theoretic properties, in particular, a clear description of their elementary classes.

Usually, we fix a class $\mathbb{C}$ and assume A and B are in $\mathbb{C}$. The most popular

In case for a given group A the elementary equivalence $A \equiv B$ implies $A \cong B$ we speak about **first order rigidity** of the group A.

Problem

What are the first order rigid linear algebraic groups?

Usually, we fix a class $\mathcal{C}$ and assume A and B are in $\mathcal{C}$. The most popular $\mathcal{C}$ is the class of finitely generated groups.

- **REDUCTIVE ALGEBRAIC GROUPS:**
- *Chevalley groups* $G(\Phi, R)$, *where* R *is a commutative ring, i.e* $SL(n, R)$, $PSL(n, R)$, $Sp(n, R)$, *etc.*
- *Twisted Chevalley groups, i.e split* $SU(n, R)$, R *is a ring with involution*
- *Extended Chevalley groups, i.e* $GL(n, R)$, $GSp(2n, R)$...
- *Isotropic groups GL(1,D), D is a division algebra*
- *Anisotropic groups* $SO(n, R)$
- **FRIENDS AN RELATIVES**
- *Kac–Moody groups*
- *Steinberg groups* $St(\Phi, R)$
- *Arithmetic lattices, i.e.,* $SL(n, Z)$

Intuition. Let $k = k_a$ be an algebraically closed field.
Source for the intuition. Affine algebraic group G_k is just an algebraic variety in the Affine space given by the set of polynomial equations equipped with group operations.
It is the same as G_k is the subgroup of $GL(n, k)$ determined by vanishing of some system of polynomial equations.
One can speak about linear algebraic groups.

For arbitrary rings the more involved point of view is applied. An algebraic group G_S is a special functor $G_S : Rings \to Groups$, and every ring R determines the group of points $G_S(R)$.

Examples: GL_n, SL_n, Sp_{2n}, SO_n **...**

At the moment we study logical properties of groups of points $G_S(R)$ but it would be highly desirable to study logical properties of the functor G_S.

Let G - linear algebraic group, k - a field.

- $R(G)$ - solvable radical of G - maximal connected closed normal solvable subgroup of G. $R(G) = 1$ - **semisimple algebraic group**.
- $R_u(G)$- unipotent radical of G - maximal connected closed normal unipotent subgroup of G. $R_u(G) = 1$ - **reductive algebraic group**.
- G - simple/almost simple, if it does not have proper closed normal/infinite normal subgroups

- $T \subset G$ is a k-torus if $T_{\bar{k}} \cong (\bar{k}^*)^r$, $r \in N$. $r \in N$ is the absolute rank of T.
- A torus $T \subset G$ splits on k if $T_k \cong (k^*)^s$.
- The k-rank of G is the maximal rank among split tori of G.
- A group G is called anisotropic if its rank is zero, that is it has no split tori.
- Examples: $SO(\mathbb{R})$,
- A group G is called isotropic if it contains a split torus.

Non-split isotropic reductive groups, while complex, are characterized by their behavior over extensions of the base field. They are not "split" in the sense that they don't have a maximal torus over the base field, and they are "isotropic" because they contain a copy of the multiplicative group.

Isomorphism classes of reductive groups

Assign: G - reductive algebraic group $\longrightarrow$ G_{an} anisotropic kernel of G.

Definition

Let T be a maximal split torus in G. Define $G_{an} = [Z_G(T), Z_G(T)]$. This anisotropic group is called anisotropic kernel of G. Moreover, $rankG_{an} = rankG - rank_k T$.

The anisotropic kernel Gan is an important ingredient of the data, together with the index of G and the type of G, that determines the k-isomorphism class of G

- Killing classifies semisimple groups and introduces the types of semisimple Lie groups: $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$.
- Chevalley shows: Split semisimple algebaic groups have a Z-structure, ("Chevalley groups"). Killing's Classification holds over algebraically closed fields.
- Borel, Tits describe the internal structure of these groups insomuch they contain unipotent elements, i.e., up to their "anisotropic kernel". It reduces the classification and structure theory to the investigation of anisotropic semisimple groups.
- 1984 Springer writes in a survey article over linear algebraic groups: *The most difficult part of a classification of reductive k-groups is the classification of semi-simple anisotropic k-groups... A complete classification of all anisotropic k-groups seems out of reach.*

Theorem (Tits-Selbach)

Let $G = G_k$ be an isotropic reductive group. Assign to G the following data (D, T, Γ, D_a) where

- *D is the root datum corresponding to simple split algebraic group $G_{\bar{k}}(\tilde{\Phi})$ of type $\tilde{\Phi}$,*
- *T is the Tits index,*
- *Γ is the subgroup of automorphisms of Dynkin diagram for $\tilde{\Phi}$, acting on Dynkin diagram of the relative root system Φ,*
- *D_a is the type of the anisotropic kernel.*

Two k-groups G_1 and G_2 are isomorphic if and only if $D_a^1 \cong D_a^2$, and the "combinatorial parameters" coincide.

Morally: Description of elementary classes of isotropic reductive groups coincides with description of isomorphism classes of such groups, For the case of arithmetic rings these classes are locally coincide.

HISTORY

Theorem (Maltsev, 1940)

Let G be a linear group. Let $H \equiv B$. Then H is linear, that is the class of all linear groups is elementary.

Is it true that a group isomorphic to linear algebraic group is algebraic? NO
Is it true that the class of all algebraic groups is elementary. NO

Theorem (Maltsev, 1961)

Two groups $G(n, K)$ *and* $G(m, K_1)$ *(where* $G = GL, SL, PGL, PSL$*,* $n, m \geq 2$*, and* K*,*K_1 *fields of characteristics 0) are elementary equivalent if and only if* $m = n$*, and fields* K *and* K_1 *are elementarily equivalent.*

Problem

For which rings and under which conditions the class of all Chevalley groups $G(\Phi, R)$ *is elementarily definable?*

Assume $rk(\Phi) \geq 2$.

- $G(\Phi, R)$, $R = K$ is an algebraically closed field (B.Zilber)
- $G(\Phi, R)$, R is a local ring (E.Bunina)
- Various matrix senisimple groups, their overgroups and subgroups over various (arithmetic) rings (Myasnikov-Sohrabi)
- $G(\Phi, R)$, R is a Dedekind ring of the arithmetic type (D.Segal-K.Tent)
- $G_{ad}(\Phi, R)$, R is an integral domain (D.Segal-K.Tent)

Main Fact
Over Dedekind rings of the arithmetic type all Chevalley groups are first order rigid. Moreover they are rich, which means first order rigid, prime, atomic, homogeneous. All their finitely generated subgroups are definable.

Main ingredients of the proof: Double centralizer theorem, description of isomorphisms of the groups, bounded generation via unipotent classes or via elementary unipotents and biinterpreation with ground rings and QFA property for rings.

The final picture is presented in the following theorem.

E.Bunina - P.Gvozdevsky (2023)

Central quotient of $G(\Phi, R)$ is the group

$$PG(\Phi, R) = G(\Phi, R)/Z(G(\Phi, R))$$

Given a group G, elements $g_1, \ldots, g_k \in G$ and a natural number m we set

$$X^{(m)}_{g_1,\ldots g_k} = \{[a, b], a, b \in G\} \cup \{a^m, a \in G\} \cup \{g_1, \ldots, g_k\}$$

Theorem (Bunina-Gvozdevsky, 2023)

Let $G(\Phi, R)$ be a Chevalley group, $rk(\Phi) \geq 2$, R an arbitrary commutative ring with 2,3 invertible for some Φ. Let $PG(\Phi, R)$ be a central quotient of $G(\Phi, R)$. Then:

- *1. The groups $PG(\Phi, R)$ are first order rigid modulo elementary equivalence of ground rings, i.e the corresponding class is elementarily definable.*
- *2. The groups $G(\Phi, R)$ which are boundedly generated with respect to some generating set $X^{(m)}_{g_1,\dots g_k}$ are first order rigid modulo elementary equivalence of ground rings, i.e the corresponding class is elementarily definable.*

Theorem (Borel, 1973)

Let k and k' be the fields. Assume that G and G_1 are simple isotropic algebraic groups.Suppose the groups of points $G(k)$ and $G_1(k')$ are isomorphic (as the abstract groups). Then the fields k and k' are isomorphic and the adjoint algebraic groups G and G_1 are isomorphic (or the algebraic groups G and G_1 are isogenic).

This means that the isomorphism classes of the groups of points of simple isotropic algebraic groups correspond to isomorphism classes of ground fields.

P.S. There are several decorations around this theorem. Borel writes "**it seems very likely that similar results are valid over anisotropic groups**".

Theorem (Gvozdevsky, 2024)

Let G_1 and G_2 be absolutely simple group schemes over rings R_1 and R_2, each with a common root datum of the geometric fibers. Let $\widetilde{\Phi_1}$ and $\widetilde{\Phi_2}$ be the corresponding absolute root systems.

- *G_2 has isotropic rank at least 2;*
- *G_2 has isotropic rank at least 2;*
- *G_1 admits an isotropic pinning with root system of rank at least 2 that has square formula, and such that the corresponding map $\widetilde{\Phi} \to \Phi$ comes from one of the Tits indexes;*
- *if $\widetilde{\Phi_1}$ is doubly laced, then $2 \in R_1^*$; if $\widetilde{\Phi}_1 = G_2$, then $6 \in R_1^*$;*
- *$2 \in R_2^*$ (independently on $\widetilde{\Phi_2}$); if $\widetilde{\Phi_2} = G_2$, then $6 \in R_2^*$.*

Theorem

Let $E_i(R_i)$ *be the elementary subgroup of* $G_i(R_i)$ $(i = 1,2)$*. Let* $\theta\colon E_1(R_1) \to E_2(R_2)$ *be the isomorphism of abstract groups. Then*

1. *If* $\widetilde{\Phi_1}$ *is not isomorphic to* $\widetilde{\Phi_2}$*, then* $\widetilde{\Phi_1} = A_3$*,* $\widetilde{\Phi_2} = B_2$*,* $R_1/M \cong \mathbb{F}_2$ *for all maximal ideals* M *in* R_1 *and* $R_2/M \cong \mathbb{F}_3$ *for all maximal ideals* M *in* R_2*.*
2. *If* $\widetilde{\Phi_1} = \widetilde{\Phi_2}$*, then there exists a ring isomorphism* $\varphi\colon R_1 \cong R_2$ *and an* R_2*-group-scheme isomorphism* $\Theta\colon {}^{\varphi}G_1 \cong G_2$ *such that* Θ *is induced by* φ*.*

Theorem (Gvozdevsky 2024)

Let X be an affine scheme of finite type over the ring S.Let J be an ultrafilter on the set I. Then for any S algebra R there is a canonical isomorphism:

$$X(\prod_J(R)) = \prod_J(X(R))$$

In particular if G absolutely simple group scheme over the ring R of rank m, then $\prod_J(G(R)) \cong G(\prod_J(R))$.

In other words ultraproduct of reductive isotropic algebraic (root graded) groups is again the group of the same type over the ultraproduct of rings. Ultrapower of anisotropic groups is again anisotropic group and so on.

Theorem (Gvozdevsky)

Let G_1 and G_2 be absolutely simple adjoint isotropic root graded groups, which satisfy conditions of the previous theorem. Suppose that for commutative rings R_1 and R_2 the groups of points $G(R_1)$ and $G(R_2)$ are elementarily equivalent. Then

- *either there are a ring S and elementary embeddings $i_1 : R_1 \to S$ and $i_2 : R_2 \to S$ such that the base changes of G_1 and G_2 to S via i_1 and i_2 are isomorphic as group schemes over S*
- *or $\widetilde{\Phi}_1 = A_3$ and $\widetilde{\Phi}_2 = B_2$. $R_1/M \cong \mathbb{F}_2$ for all maximal ideals M in R_1 and $R_2/M \cong \mathbb{F}_3$ for all maximal ideals M in R_2.*

Theorem

Let G_1 be absolutely simple adjoint isotropic root graded group. Suppose that for commutative ring R_1 the groups $G_1(R_1)$ and H are elementarily equivalent. Then there are rings R_2 and S, elementary embeddings $i_1 : R_1 \to S$ and $i_2 : R_2 \to S$, and a group scheme G_2 over R_2 such that $H \cong G_2(R_2)$ and the base changes of G_1 and G_2 to S via i_1 and i_2 are isomorphic as group schemes over S.

Theorem

Let G, R be absolutely simple adjoint isotropic root graded groups, subject to all necessary conditions. Suppose that G as a group scheme has a form over a subring of R, which is either $\mathbb{Z}$, or quotient of $\mathbb{Z}$, or localization of $\mathbb{Z}$. Then $G(R)$ is regularly biinterpretable with R.