On the Cremona dimension of a finite group

Zinovy Reichstein
University of British Columbia, Canada
Joint work with Giulio Bresciani and
Angelo Vistoli (Scuola Normale Superiore, Italy)

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The classical Cremona group

Let X be a smooth complex projective variety. We denote by $\mathrm{Bir}(X)$ the group of birational transformations $X \dashrightarrow X$.

The plane Cremona groups $\mathrm{Cr}_n = \mathrm{Bir}(\mathbb{P}^n)$ are objects of classical interest. Cr_2 was studied in the 19th century by Max Noether, Guido Castelnuovo and many others. Note that these are not algebraic groups (they are infinite-dimensional), though they share some common features with algebraic groups.

Our primany interest will be in the finite subgroups of these groups. In 2009 Igor Dolgachev and Vasily Iskovskikh, classified the conjugacy classes of finite subgroups of Cr_2. Their classification is rather complicated; extending it to higher Cremona groups seems hopeless.

The cremona dimension

In this talk we will take a different approach. Rather than fixing n and asking for a complete list of finite subgroups of Cr_n, we will fix a finite group G and ask for the smallest n such that G embeds into Cr_n. We will call this number the cremona dimension of G and denote it by $\mathrm{crdim}(G)$. (Note the small "c" in "crdim"!) Equivalently, $\mathrm{crdim}(G)$ is the smallest integer such that there exists a faithful G-action on a rational n-dimensional variety.

Jean-Pierre Serre asked whether or not there exists a finite group such that $\mathrm{crdim}(G) > 3$ or more generally such that $\mathrm{crdim}(G) > n$ for a fixed n. He expected a positive answer for any n.

The Jordan property

Let Γ be an abstract group. We say that Γ has the Jordan property if there exists an integer j such that every finite subgroup $S \leqslant \Gamma$ contains an abelian normal (in S) subgroup A of index $[S : A] \leqslant j$. The constant j is called a Jordan constant for Γ. It is not unique.
In 1878 Camile Jordan showed that the general linear group $\mathrm{GL}_n(\mathbb{C})$ has the Jordan property. In 2009 Serre showed that Cr_2 has the Jordan property and asked whether or not Cr_n does as well, for every $n \geqslant 2$. If true, this would immediately imply that for any given n there are only finitely many finite simple groups G such that $\mathrm{crdim}(G) \leqslant n$.

In 2016 Yury Prokhorov and Konstantin Shramov showed that the Jordan property of Cr_n follows from the Borisov-Alexeev-Borisov (BAB) conjecture for n-folds.

BAB Conjecture: The Fano varieties of fixed dimension n with terminal singularities are bounded, i.e., fit into finitely many algebraic families.

Alexander Borisov proved the BAB conjecture in dimension 3 in 1996. Thus Cr_3 has the Jordan Property.

Birkar's theorem and explicit bounds

Soon after the work of Prokhorov and Shramov, the BAB conjecture was proved in full generality by Caucher Birkar. As a corollary, Cr_n has the Jordan property for every n, confirming Serre's expectation. In particular,

$$\lim_{n\to\infty} \mathrm{crdim}(S_n) = \infty,$$

and the number of finite simple groups of bounded Cremona dimension is finite.

Unfortunately a specific Jordan constant $j(n)$ for Cr_n is out of reach. It is not easy to exhibit one even for $n = 2$ and 3: Prokhorov and Shramov showed that $j(3)$ can be taken to be $107\,495\,424$.

So the problem of giving explicit examples of finite groups of Cremona dimension at least 5 remained open.

The Cremona dimension

All of the above results by Prokhorov-Shramov and Birkar concern groups of birational isomorphisms automorphisms $\mathrm{Bir}(X)$, where X is a rationally connected variety.

Recall that X is called rationally connected if any two points on X can be connected by a chain of rational curves. This larger class of varieties turns out to be easier to work with than the class of rational or unirational varieties.

In particular, Prokhorov-Shramov and Birkar showed that for rationally connected varieties of fixed dimension n, the groups $\mathrm{Bir}(X)$ are uniformly Jordan. In other words, they all have the Jordan property with the same constant $j(n)$, depending only on $n = \dim(X)$.

From now on, we will work with $\mathrm{Crdim}(G)$ instead of $\mathrm{crdim}(G)$. Here G is a finite group and $\mathrm{Crdim}(G)$ is the minimal dimension of a rationally connected variety X which admits a faithful action of G.

Clearly, $\mathrm{crdim}(G) \geqslant \mathrm{Crdim}(G)$, so any lower bound of $\mathrm{Crdim}(G)$ will also be a lower bound on $\mathrm{crdim}(G)$.

Elementary properties of the Cremona dimension

Lemma:

(a) If H is a subgroup of G, then $\mathrm{Crdim}(G) \geqslant \mathrm{Crdim}(H)$.

(b) If N is a normal subgroup of G, and $\overline{G} = G/N$, them $\mathrm{Crdim}(G) \geqslant \mathrm{Crdim}(\overline{G})$.

(c) $\mathrm{Crdim}(G_1 \times G_2) \leqslant \mathrm{Crdim}(G_1) + \mathrm{Crdim}(G_2)$.

Proof: (a) If G acts faithfully on a rationally connected variety X, then $H \leqslant G$ acts faithfully on the same X.

(b) If G acts faithfully on a rationally connected variety X, then $\overline{G} := G/N$ acts faithfully on $\overline{X} := X/N$. Moreover, $\overline{X}$ is again rationally connected and $\dim(\overline{X}) = \dim(X)$.

(c) If G_1 acts faithfully on X_1 and G_2 acts faithfully on X_2, then $G_1 \times G_2$ acts faithfully on $X_1 \times X_2$.

Haution's fixed point theorem

Theorem (O. Haution, 2018). Let p be a prime, G a finite p-group acting on a smooth projective variety X of dimension $< p-1$. Assume that $p \nmid \chi(X, \mathcal{O}_X)$. Then $X^G \neq \varnothing$.

If X is rationally connected then $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$, hence $\chi(X, \mathcal{O}_X) = 1$, and the assumption that $p \nmid \chi(X, \mathcal{O}_X)$ is satisfied for every prime p.

Corollary (J. Xu). Let p be a prime, X a smooth projective variety such that $p \nmid \chi(X, \mathcal{O}_X)$, G a finite p-subgroup acting on X. If $\dim X < p-1$, then G is abelian of rank at most $\dim X$.

Proof sketch: By Haution's theorem, G has a fixed point on X. The G- representation on the tangent space at that point is faithful, and a p-group with a faithful representation V with $\dim V \leqslant p-1$ is abelian of rank at most $\dim V$.

Consequences of Haution's theorem

This immediately implies the following:

- A non-abelian p-group has Cremona dimension at least $p-1$.
- An abelian p-group of rank r has Cremona dimension at least $\min(r, p-1)$.

Previously Prokhorov and Shramov had shown that a non-abelian p-group has Cremona dimension at least 4 if $p \geqslant 17$.

The symmetric group S_n contains a non-abelian p-group if and only if $n \geqslant p^2$; hence if p is a prime with $p \leqslant \sqrt{n}$ we have $\mathrm{Crdim}(S_n) \geqslant p-1$. By Bertrand's postulate (proved by Chebyshev), there is a prime p with $\lfloor \sqrt{n}/2 \rfloor < p < \sqrt{n}$. Thus

$$\lfloor \sqrt{n}/2 \rfloor \leqslant \mathrm{Crdim}(\mathrm{S}_n).$$

The best known upper bound is roughly $\mathrm{Crdim}(\mathrm{S}_n) \leqslant n - \sqrt{2(n-1)}$.

Essential dimension

A limitation of Haution's theorem is the assumption that $\dim X < p-1$; for example, it says nothing about 2-subgroups of $\mathrm{Bir}(X)$. In this talk I would like to present new (stronger) lower bounds on $\mathrm{Crdim}(G)$, where G is a non-abelian p-group.
To motivate our results, I will first discuss a different numerical invariant of a finite group. The essential dimension of a finite group G is the minimal dimension of an algebraic variety X such that X admits a faithful *linearizable action* of G. Here an action of G on X is called linearizable if there exists a dominant linear G-equivariant map $V \dashrightarrow X$, where V is a vector space equipped with a faithful linear action of G. Clearly any linearizable X is unirational; hence,

$$\mathrm{ed}(G) \geqslant \mathrm{Crdim}(G).$$

An equivalent definition of $\mathrm{ed}(G)$ is as follows. Start with a faithful linear representation V (doesn't matter which one). Then $\mathrm{ed}(G)$ is the minimal dimension of $f(V)$, where the minimum is taken over all G-equivariant rational maps $f\colon V \dashrightarrow V$ with faithful G-action on $f(V)$.

A lower bound on the essential dimension: Version 1

Let G be a finite p-group. Denote the center of G by μ. Assume that μ is cyclic of order p^r. The exact sequence

$$1 \to \mu \to G \to \overline{G} \to 1.$$

gives rise to the connecting morphism $\partial_K \colon H^1(K, \overline{G}) \to H^2(K, \mu)$. I will assume that K contains our base field $\mathbb{C}$. In this case we can identify $H^2(K, \mu)$ with the p^r-torsion subgroup of the Brauer group $\mathrm{Br}(K)$. Let $\mathrm{Ind}(G)$ be the largest index of the Brauer class $\partial_K(\tau)$, as K containing $\mathbb{C}$ and $\tau \colon T \to \mathrm{Spec}(K)$ ranges over all $\overline{G}$-torsors over K.

Theorem 1: $\mathrm{ed}(G) \geqslant \mathrm{Ind}(G)$.

I want to outline a proof of this result due to Mathieu Florence, then use a variant of this argument to bound the Cremona dimension of G. Florence's argument relies on

Karpenko's Incompressibility Theorem: Let P be a Brauer-Severi variety over a field K of index p^n, where p is a prime. Then for any rational map $\phi \colon P \dashrightarrow P$, the dimension of the image $\phi(P)$ is $\geqslant p^n - 1$.

Proof of Theorem 1. Step 1: homogenization

Since μ is cyclic, there exists a faithful irreducible representation V of G such that elements of μ act on V as scalars. Recall that $\mathrm{ed}(G)$ is the minimal value of $\dim\, f(V)$, where the minimum is taken over all G-equivariant rational maps $f\colon V \dashrightarrow V$ such that the G-action on $f(V)$ is faithful. Choose a G-equivaiant rational map $f\colon V \dashrightarrow V$ as above such that $\dim\, f(V)$ takes on its minimal value $\mathrm{ed}(G)$.

Step 1 (Homogenization): f can be chosen to be homogeneous. In other words, there is an integer d (called the degree of f) such that $f(t \cdot v) = t^d f(v)$ for every $v \in V$ and $t \in \mathbb{C}$. Since f is homogeneous, it descends to a G-equivariant rational map $\overline{f}\colon \mathbb{P}(V) \dashrightarrow \mathbb{P}(V)$. Note that elements of μ act trivially on $\mathbb{P}(V)$. Hence, the G-action on V descends to a $\overline{G} = G/\mu$-action on $\mathbb{P}(V)$.

Step 2: Twisting

Step 2. Recall that for any field K containing $\mathbb{C}$, we have a connecting homomorphism $\partial_K \colon H^1(K, \overline{G}) \to H^2(K, \mu)$. It can be described explicitly as follows.

Elements of $H^1(K, \overline{G})$ are represented by $\tau \colon \overline{G}$-torsors $T \to \mathrm{Spec}(K)$.

Let ${}^{\tau}\mathbb{P}(V)$ be the twist of $\mathbb{P}(V)$ by τ. In concrete terms, ${}^{\tau}\mathbb{P}(V)$ is the quotient of $T \times_K \mathbb{P}(V)$ by the (free) diagonal action of $\overline{G}$. Note that ${}^{\tau}\mathbb{P}(V)$ does not carry a G-action, it is just a variety over K. (The G-action is "used up" in the twisting construction.)

The twist ${}^{\tau}\mathbb{P}(V)$ is a Brauer-Severi variety (over K). Indeed, over the algebraic closure of K, T splits, and ${}^{\tau}\mathbb{P}(V)$ becomes isomorphic to $\mathbb{P}(V)$.

The Brauer-Severi variety represents a Brauer class over K. This Brauer class is exactly $\partial_K(\tau)$.

Step 3: Karpenko incompressibility

Since the map $\overline{f}\colon \mathbb{P}(V) \dashrightarrow \mathbb{P}(V)$ is $\overline{G}$-equivariant, it gives rise to a rational map

$$^{\tau}\overline{f}\colon {}^{\tau}\mathbb{P}(V) \dashrightarrow {}^{\tau}\mathbb{P}(V).$$

By Karpenko's Incompressibility Theorem, $\dim_K \mathrm{Im}({}^{\tau}\,\overline{f}) \geqslant \mathrm{Ind}(\partial_K(\tau)) - 1$. Now observe that the image of ${}^{\tau}\,\overline{f}$ has the same dimension as the image of $\overline{f}$. (To see this, pass to the algebraic closure of K.) Thus

$$\dim_{\mathbb{C}} \mathrm{Im}(\overline{f}) \geqslant \mathrm{Ind}(\partial_K(\tau)) - 1.$$

Choosing K and $\tau\colon T \to \mathrm{Spec}(K)$ so that $\mathrm{Ind}(\partial_K(\tau))$ takes on its largest possible value, we obtain

$$\dim_{\mathbb{C}} \mathrm{Im}(\overline{f}) \geqslant \mathrm{Ind}(G) - 1.$$

Now recall that $f\colon V \dashrightarrow V$ is a homogeneous rational map and $\overline{f}\colon \mathbb{P}(V) \dashrightarrow \mathbb{P}(V)$ is the induced rational map of projective spaces. Thus $\mathrm{Im}(f)$ is an affine code over $\mathrm{Im}(\overline{f})$, and consequently, $\mathrm{ed}(G) = \dim_{\mathbb{C}} \mathrm{Im}(f) = \dim_{\mathbb{C}} \mathrm{Im}(\overline{f}) + 1 \geqslant \mathrm{Ind}(G)$, as desired.

A lower bound on the essential dimension, Version 2

Now let G be an arbitrary finite p-group with center μ (not necessarily cyclic). I claim that the lower bound on $\mathrm{ed}(G)$ from Theorem 1 continues to hold in this setting, provided that we modify the definition of $\mathrm{Ind}(G)$.

Theorem 2: $\mathrm{ed}(G) \geqslant \mathrm{Ind}(G)$.

Once again, we consider the connecting homomorphism $\partial_K \colon H^1(K, \overline{G}) \to H^2(K, \mu)$ associated to the central exact sequence

$$1 \to \mu \to G \to \overline{G} \to 1.$$

Since μ is not cyclic, we can no longer identify $\alpha = \partial_K(\tau)$ in $H^2(K, \mu)$ with a Brauer class. However, any character $\chi \colon \mu \to \mathbb{G}_m$, gives rise to a map $\chi_* \colon H^2(K, \mu) \to H^2(K, \mathbb{G}_m)$ and thus a Brauer class $\chi_*(\alpha)$ in $H^2(K, \mathbb{G}_m) = \mathrm{Br}(K)$. As χ ranges over $\mu^* = \mathrm{Hom}(\mu, \mathbb{G}_m)$, the Brauer classes $\chi_*(\alpha)$ form a finite p-subgroup A of $\mathrm{Br}(K)$.

The index of a class in $H^2(K, \mu)$

The index $\mathrm{Ind}(A)$ associated to a finite subgroup A of $\mathrm{Br}(K)$ is defined as the minimal value of

$$\mathrm{Ind}(a_1) + \ldots + \mathrm{Ind}(a_r)$$

where the minimum is taken over all generating sets $a_1, \ldots, a_r$ of A.
Back to our setting: consider the connecting homomorphism $\partial_K \colon H^1(K, \overline{G}) \to H^2(K, \mu)$ associated to the central exact sequence

$$1 \to \mu \to G \to \overline{G} \to 1.$$

For any $\tau \in H^1(K, \overline{G})$, we define the index of $\alpha = \partial_K(\tau) \in H^2(K, \mu)$ as $\mathrm{Ind}(A)$, where A is the finite subgroup of $\mathrm{Br}(K) = H^2(K, \mathbb{G}_m)$ consisting of elements of the form $\chi_*(\alpha)$, as $\chi \colon \mu \to \mathbb{G}_m$ ranges over the characters of μ. Finally, we define $\mathrm{Ind}(G)$ as the maximal value of the index of $\alpha = \partial_K(\tau)$, as τ ranges over $H^1(K, \overline{G})$. Theorem 2 then asserts that $\mathrm{ed}(G) \geqslant \mathrm{Ind}(G)$.

Proof of Theorem 2: Step 1

The proof of Theorem 2 is similar to the proof of Theorem 1. We choose K, $\tau \in H^1(K, \overline{G})$ and $\chi_1, \ldots, \chi_r : \mu \to \mathbb{G}_m$ which give rise to the maximal index. In other words,

$$\mathrm{Ind}(G) = \mathrm{Ind}((\chi_1)_*(\partial_K(\tau)) + \ldots + \mathrm{Ind}(\chi_r)_*(\partial_K(\tau)).$$

We now choose irreducible representations $V_1, \ldots, V_r$ such of G that μ acts on V_i via scalar multiplication by χ_i. Set $V = V_1 \times \ldots \times V_r$. Recall that $\mathrm{ed}(G)$ is the minimal value of $\dim \mathrm{Im}(f)$, as f ranges over the G-equivariant maps $V \dashrightarrow V$ such that G acts faithfully on $\mathrm{Im}(f)$.

We now proceed in three steps, as before:

Step 1: Multi-homogenization. f can be chosen to be multi-homogeneous with respect to $V_1, \ldots, V_r$. This modification of Florence's homogenization argument is due to Roland Lötscher.

Proof of Theorem 2: Steps 2 and 3

Step 2: A multi-homogeneous $f\colon V_1 \times \ldots \times V_r \dashrightarrow V_1 \times \ldots \times V_r$ induces a G-equivariant map

$$\overline{f}\colon \mathbb{P}(V_1) \times \ldots \times \mathbb{P}(V_r) \dashrightarrow \mathbb{P}(V_1) \times \ldots \times \mathbb{P}(V_r).$$

We twist this map by τ,

$$^{\tau}\,\overline{f}\colon {}^{\tau}\,\mathbb{P}(V_1) \times \ldots \times {}^{\tau}\,\mathbb{P}(V_r) \dashrightarrow {}^{\tau}\,\mathbb{P}(V_1) \times \ldots \times {}^{\tau}\,\mathbb{P}(V_r).$$

Step 3: We apply the following to this map. The rest of the argument is the same as before.

Karpenko Incompressibility, Version 2, due to Karpenko and Merkurjev: Let $P_1, \ldots, P_r$ be Brauer-Severi varieties, whose classes form a p-subgroup A in $\mathrm{Br}(K)$. Then for any rational map

$$\phi\colon P_1 \times \ldots \times P_r \dashrightarrow P_1 \times \ldots \times P_r$$

defined over K, $\dim_K \mathrm{Im}(\phi) \geqslant \mathrm{Ind}(A) - r$.

A lower bound on the Cremona dimension

It turns out that a similar argument can be used to prove a lower bound on the Cremona dimension. The main difference is that the index gets replaced by the exponent. Once again, assume that G is a finite p-group. Denote its center by μ and consider the connecting homomorphism $\partial_K \colon H^1(K, \overline{G}) \to H^2(K, \mu)$ induced by the exact sequence

$$1 \to \mu \to G \to \overline{G} \to 1.$$

As we discussed before, every $\tau \in H^1(K, \overline{G})$ gives rise to a finite p-subgroup $A(\tau) \leqslant \mathrm{Br}(K)$ consisting of elements of the form $\chi_*(\partial_K(\tau)) \in H^2(K, \mathbb{G}_m)$, as χ ranges over the characters $\mu \to \mathbb{G}_m$.

For a finite p-subgroup A of $\mathrm{Br}(K)$, we define $\exp(A)$ as the minimal value of

$$\exp(a_1) + \ldots + \exp(a_r)$$

as $a_1, \ldots, a_r$ range over the generating sets of A.

Let us now define $\gamma(G)$ to be the maximal value of $\exp(A(\tau)) - r$, as K ranges over all fields containing $\mathbb{C}$ and τ ranges over $H^1(K, \mathbb{G})$.

Theorem 3: $\mathrm{Crdim}(G) \geqslant \gamma(G)$.

A version of Karpenko incompressibility due to Bresciani and Vistoli

The proof of Theorem 3 proceeds along the same lines we outlined earlier. The main new ingrediend is the following.

Karpenko Incompressibility Theorem, Version 3, due to Bresciani and Vistoli.
Let $P_1, \dots, P_r$ be Brauer-Severi varieties over a field K and A be the subgroup of $\mathrm{Br}(K)$ generated by their classes. Assume that A is a p-group (finite and abelian, always). Let Y be a geometrically rationally connected variety over K. (In other words, over the algebraic closure of K, Y becomes rationally connected.)

Then for any rational map $\phi\colon Y \dashrightarrow P_1 \times \dots \times P_r$, defined over K, $\dim_K \mathrm{Im}(\phi) \geqslant \gamma(A)$.

Proof of Theorem 3

To deduce Theorem 3 from Karpenko incompressibility, assume X is a rationally connected variety which admits a faithful action of a finite group G. Out goal is to show that $\dim(X) \geqslant \gamma(G)$. Choose $\tau \in H^1(K, \overline{G})$ and a generating set $\chi_1, \ldots \chi_r$ of μ^* such that

$$\gamma(A(\tau)) = \exp(A(\tau) - r = \exp\left((\chi_1)_*(\partial_K(\tau))\right) + \ldots + \exp\left((\chi_r)_*(\partial_K(\tau))\right) - r$$

gives rise to an isomorphism

$$\gamma(G) = \exp(A(\mu)) - r$$

Next choose irreducible representations $V_1, \ldots, V_r$ such that μ acts by χ_i on each V_i. Once again, an elementary group-theoretic argument shows that the G-action on $V = V_1 \times \ldots \times V_r$ is faithful.
Now there exists a G-equivariant rational map $f \colon X \dashrightarrow V$ such that the G-action on $f(X)$ is faithful. (This follows from the fact that V is versal.) We now proceed in the same way as before, with some modifications.

Proof of Theorem 3. Step 1: Homogenization

Since X is not a vector space, there is no way to homogenize (or multi-homogenize) f. So, there is no Step 1 here. We ultimately pay a price for skipping this step by having $\mathrm{Crdim}(G) \geqslant \exp(A(\tau)) - r$ in the final formula, rather than $\mathrm{crdim}(G) \geqslant \exp(A(\tau))$.

Proof of Theorem 3. Steps 2 and 3.

Step 2: The G-equivariant map $f\colon X \dashrightarrow V_1 \times \ldots \times V_r$ descends to

$$\overline{f}\colon X/\mu \dashrightarrow \mathbb{P}(V_1) \times \ldots \times \mathbb{P}(V_r)$$

We now twist both sides by τ to get a rational map

$$^{\tau}\,\overline{f}\colon {}^{\tau}\,(X/\mu) \dashrightarrow {}^{\tau}\,\mathbb{P}(V_1) \times \ldots \times {}^{\tau}\,\mathbb{P}(V_r).$$

Step 3: Note that since X is rationally connected, so is X/μ. Applying Karpenko Incompressibility, Version 3 to $^{\tau}\,\overline{f}$, with $Y = {}^{\tau}\,(X/\mu)$, we conclude that

$$\dim_{\mathbb{C}} \operatorname{Im}(f) \geqslant \dim_{\mathbb{C}} \operatorname{Im}(\overline{f}) \geqslant \dim_K \operatorname{Im}({}^{\tau}\,\overline{f}) \geqslant \exp(A(\tau)) - r = \gamma(G).$$

A group-theoretic interpretation

The invariants $\mathrm{Ind}(G)$ and $\gamma(G)$ have been defined in cohomological terms. Here is a direct group-theoretic description:

Proposition: Let G be a finite p-group and μ be the center of G. Then

(a) (Karpenko and Merkurjev) $\mathrm{Ind}(G)$ is the minimal dimension of a faithful representation of G.

(b) Write $\mu \cap [G, G]$ as $\mathbb{Z}/p^{e_1} \times \ldots \mathbb{Z}/p^{e_r}\mathbb{Z}$. Then $\gamma(G) = p_1^{e_1} + \ldots + p_r^{e_r} - r$.

Example: A group of order p^3

Fix a prime p and let E_1 be the non-commutative group of order p^3 generated by elements

$$x, y \text{ and } z$$

of order p. Here z is central, and $xy = zyx$. Note that E_1 will be the first in a sequence of extra-special groups $\{E_n \mid n = 1, 2, \ldots\}$, where $|E_n| = p^{2n+1}$. We will consider the groups E_n for $n \geqslant 2$ later on. Note also that E_1 depends on p.

Theorem 4: $\mathrm{Crdim}(E_1^s) = s(p-1)$ for any integer $s \geqslant 1$.

Proof: The inequality $\mathrm{Crdim}(E_1^s) \geqslant s(p-1)$ is an easy consequence of Theorem 3. Indeed, set $G = E_1^s$. Then $\mu \cap [G, G] \simeq (\mathbb{Z}/p\mathbb{Z})^s$ and thus $\gamma(G) = sp - s$. We conclude that

$$\mathrm{Crdim}(E_1^s) \geqslant \gamma(E_1^s) = s(p-1).$$

Proof of the inequality $\mathrm{Crdim}(E_1^s) \leqslant s(p-1)$

It remains to show that $\mathrm{Crdim}(E_1^s) \leqslant s(p-1)$. As we saw earlier, Cremona dimension is sub-additive: in particular, $\mathrm{Crdim}(E_1^s) \leqslant s\,\mathrm{Crdim}(E_1)$. Thus it suffices to show that $\mathrm{Crdim}(E_1) \leqslant p-1$.

First assume $p=2$. In this case, E_1 is the dihedral group of order 8. This group acts faithfully on $\mathbb{P}^1$; hence, $\mathrm{Crdim}(E_1)=1$, as desired.

Now assume $p>2$. Let V be the p-dimensional natural representation of E_1 sending

x_1 to the diagonal matrix $\mathrm{diag}(1,\zeta,\zeta^2,...,\zeta^{p-1})$,

y_1 to a permutation matrix cyclicly permuting the standard basis vectors $e_1,\dots,e_p$, and

z to ζI_p, where ζ is a primitive pth root of unity and I_p is the $p\times p$ identity matrix.

Now consider the affine hypersurface X in V given by $x_1x_2...x_p=1$. It is easy to see that X is rational and that E_1 acts faithfully on X (here we use the assumption that p is odd). Thus $\mathrm{Crdim}(E_1) \leqslant \dim(X) = p-1$. This completes the proof of Theorem 4.

The Cremona dimension of the symmetric group S_n

S_n contains the product of $\lfloor n/4 \rfloor$ copies of S_4, and each S_4 contains a copy of the dihedral group D_8, which is the same as E_1 for $p = 2$.
Using Theorem 5, we see that

$$\mathrm{Crdim}(\mathrm{S}_n) \geqslant \mathrm{Crdim}(\mathrm{S}_4^{\lfloor n/4 \rfloor}) = \mathrm{Crdim}(\mathrm{D}_8^{\lfloor n/4 \rfloor}) = \lfloor n/4 \rfloor \cdot (2-1) = \lfloor n/4 \rfloor .$$

Remarks: (1) This inequality has also been proved by János Kollár and Ziquan Zhuang by a different method.

(2) As mentioned above, the best known upper bound is roughly

$$\mathrm{Crdim}(\mathrm{S}_n) \leqslant n - \sqrt{2(n-1)}.$$

(3) The best known bounds on the essential dimension of S_n are

$$\lfloor (n+1)/2 \rfloor \leqslant \mathrm{ed}(\mathrm{S}_n) \leqslant n-3$$

for any $n \geqslant 7$.

Extra-special groups

Let E_n be the extra-special group of order p^{2n+1} generated by elements

$$x_1, y_1, ..., x_n, y_n \text{ and } z$$

of order p; any two of these elements commute, except that $x_i y_i = z y_i x_i$ for each $i = 1, \ldots, n$.

Theorem 6: $p - 1 \leqslant \mathrm{Crdim}(E_n) \leqslant n(p-1)$.

The lower bound follows from Haution's Theorem: $\mathrm{Crdim}(G) \geqslant p - 1$ for any non-abelian p-group G. Another proof is as follows: since E_n contains a copy of E_1 we have $\mathrm{Crdim}(E_n) \geqslant \mathrm{Crdim}(E_1) = p - 1$, where the last equality is a special case of Theorem 4.

To prove the upper bound, observe that E_n is a quotient of the product $E_1^n := E_1 \times \ldots \times E_1$ (n times). Hence, by one of the elementary properties of the Cremona dimension,

$$\mathrm{Crdim}(E_n) \leqslant \mathrm{Crdim}(E_1^n) = n(p-1).$$

A better lower bound on $\mathrm{Crdim}(G)$?

For $G = E_n$, there is a big gap between $\mathrm{Ind}(E_n) = p^n$ and $\gamma(E_n) = p - 1$ and consequently, a big gap between $\mathrm{ed}(E_n) = p^n$ and $\mathrm{Crdim}(E_n) \leqslant n(p-1)$. This is not a deficiency of our methods; the Cremona dimension of E_n just happens to be a lot lower than the essential dimension.

On the other hand, the lower bound $\mathrm{Crdim}(E_n) \geqslant p - 1$ does not seem optimal; in particular, it "should" dependend of n.

It is just natural to look for a better lower bound on $\mathrm{crdim}(G)$, which depends on $\mathrm{Ind}(G)$ as well as $\gamma(G)$. Here G is a finite p-group. So far we have only figured it out what this better bound should be in the case, where μ = center of G is cyclic.

A better lower bound on $\mathrm{Crdim}(G)$, continued

Assume G is a finite p-group such that the center μ of G is cyclic. Recall that the exact sequence

$$1 \to \mu \to G \to \overline{G} \to 1.$$

gives rise to the connecting morphism $\partial_K \colon H^1(K, \overline{G}) \to H^2(K, \mu)$. Recall also that $H^2(K, \mu)$ is naturally identified with a subgroup of the Brauer group $\mathrm{Br}(K)$.
Denote the maximal index and the maximal exponent of the Brauer class $\partial_K(\tau)$ by p^i and p^e, respectively. In concrete terms p^e can also be thought of as the order of the cyclic group $\mu \cap [G, G]$ and p^i as the minimal dimension of a faithful representation of G.

Theorem 7: If $i \leqslant pe$, then $\mathrm{Crdim}(G) \geqslant \left(1 + \lfloor \dfrac{i-1}{e} \rfloor\right)(p^e - 1)$.

A better lower bound on $\mathrm{Crdim}(E_n)$

Let us now apply Theorem 7 to the extra-special group E_n of order p^{2n+1} we considered earlier. Here $i = n$ and $e = 1$. Substituting these values into Theorem 7 and remembering the upper bound $\mathrm{Crdim}(E_n) \leqslant n(p-1)$ proved earlier, we obtain.

Theorem 8: If $p \geqslant n$, then $\mathrm{Crdim}(E_n) = n(p-1)$.

We have a better lower bound than $p-1$ in the case, where $p < n$, as well though it does not quite meet the upper bound of $n(p-1)$.