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Linear Algebraic Groups vs Automorphism Groups

Ivan Arzhantsev

In this talk we discuss the similarities and differences between linear algebraic groups and automorphism groups of affine algebraic varieties. We are especially interested in the group of special automorphisms $SAut(X)$, i.e., the normal subgroup of $Aut(X)$ generated by all one- parameter additive subgroups. We consider the Tits Alternative and the flexibility property for affine varieties. The group $SAut(X)$ acts in the regular locus of a flexible variety X infinitely transitively, that is, any finite collection of smooth points can be sent to any finite collection of smooth points of the same cardinality. Using flexibility, we show that every non-degenerate toric variety, every homogeneous space of a semisimple group, and every variety covered by affine spaces admits a surjective morphism from an affine space. Applying the ellipticity property introduced by Mikhail Gromov in 1989, we prove that a complete algebraic variety X is an image of an affine space if and only if X is unirational.

This is a joint result with Shulim Kaliman and Mikhail Zaidenberg.

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Gröbner-Shirshov bases, Dehn's algorithm and small cancellation rings

Agatha Atkarskaya

In Group Theory there exists a class of problems on constructing groups with exotic properties, e.g., infinite Burnside groups, Tarskii Monster etc.. There exists a universal approach for these problems which is called "Iterated small cancellation theory". Its essence is that the desired group is presented as a direct limit of hyperbolic groups (or even stronger small cancellation groups). In Ring Theory there exists a similar class of problems on constructing algebras with exotic properties. Particularly, we are interested in the following old problem: does there exist a division algebra infinite dimensional over its center with finitely generated multiplicative group (posed by Kaplansky, Lvov and Latyshev in 1970s). For algebras so far there is no uniform approach to construct such "monster" objects. In papers [1, 2, 3] we made a significant progress towards such an approach. Namely, we introduced new techniques which allow to work with ring relations using the idea of small cancellation form Group Theory, gave a definition of small cancellation rings, and described their properties and further applications.

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Ideas for a radical reform of mathematics education, après Arnold, Gelfand, and Vavilov

Alexandre Borovik

Introduction

I continue the discussion started by Nikolai Vavilov in his remarkable paper of 2021 (with Vladimir Khalin and Alexander Yurkov) *The skies are falling: Mathematics for non-mathematicians* [10] supported by their texbook [9]. It is about the crisis in mathematics education and the urgent need for a deep reform. I will invoke views on possible changes from other mathematicians, first of all, Vladimir Arnold and Israel Gelfand.

But my assessment is more stark: the mainstream mathematics education *for everyone* is dying. Its fate appears to be sealed by the emergence of AI with its promise that a smartphone will answer all mathematical questions which a person would ever encounter in his/her everyday life. This is happening for reasons beyond our control: the change of the role of mathematics in the very heart of the economy—in division of labour [3, 4]. I do not know how to save or reform the mainstream mathematics education. Instead, I focus on development of a much more narrow stream of academically selective education, *mathematics for makers* [4] build on the principles of *Deep Education*, starting in primary school [3, Sections 8–10]:

> Mathematics education in which every stage, starting from pre-school, is designed to fit the individual cognitive profile of the child and to serve as propaedeutics of his/her even deeper study of mathematics at later stages of education—including transition to higher level of abstraction and changes of conceptual frameworks.

This means running a network of mathematical circles, olympiads, Sunday, Winter, Easter, Summer Schools, etc. for identifying children who must be saved from their schools and directed to mathematical classes, schools, boarding schools of the *Deep Stream*. This is a colossal task, and it cannot be done without an active participation of the professional mathematical community. We have to realise that we have to ensure survival of our community and our culture. It is worth to remember a message from the times when a certain community of faith was fighting for its survival:

> *Then saith he unto his disciples, The harvest truly is plentious, but the labourers are few.* (Matthew 9:37 KJV)

1. Deep Stream: Arithmetic + Algorithmic + Algebra

With the administrative structure of the Deep Stream so loose, some shared understanding of its purpose and possible curricula is paramount. I emphasize: this is not about adapting the 20th century mathematics to the 21st century. As outlined in [6, 10], it means the 21st century mathematics education for the 21st century mathematics.

1.1. Arithmetic and Algorithmic

What I suggest in my talk is a fragment of one of possible curricula based on merging mathematics with computer science / programming, a course *Arithmetic, Algorithmic, Algebra* (AAA) which should start in primary school as soon as kids can read and write/type. Its first principle:

> From a relatively early stage of the course, Learner's answer to an arithmetic or algebraic problem (including mathematical problems described as 'real life' problems) should be an algorithm (and later—executable computer code developed, almost in its entirety, by Learner—without use of standard packages such as MATHEMATICA)—which
> - solves *all* problems of the same type;
> - helps to check, analyse, and generalise the solution.

At the earliest stage, this is achievable by applying the *questions procedure* to 'word problems' [5, Sections 3 and 4]: a sequence of questions produced in the solution is already an algorithm written in plain language. Together with attention to *named numbers* (future *typed variables* of coding) [5, Sections 3 and 4] this already meets the requirements.

I stick to two principles formulated in 1947 by Igor Arnold[1] [5, Section 1]:

> **1.** *Teaching arithmetic involves, as a key component, the development of an ability to negotiate situations whose concrete natures represent very different relations between magnitudes and quantities.*'

In modern mathematical language this means mapping the structures and relations of the real world into operations and relations of arithmetic.

> **2.** *The difference between the "arithmetic" approach to solving problems and the algebraic one is, primarily the need to make a concrete and*

[1]Igor Arnold was the father of Vladimir Arnold, who republished his paper and endorsed it in a touching foreword [1].

> *sensible interpretation of all the values which are used and/or which appear at any stage of the discourse.*

Again, it means full use of "functorial" properties of the mapping from the real world to numbers and other mathematical structures, and development of depth and functionality of thinking within a limited language. My favourite simile is the *Essential English* method developed in 1930s by West, Palmer, and Faucett for teaching English as a second language, which was aimed at achieving *full language fluency* within a *limited*, but *functional vocabulary.*

1.2. Algebra

Algebra should be introduced simultaneously with switch from verbal algorithms to computer programming.

The content of the algebra course should change and reflect the demands of computer programming, and include, for example, Boolean algebra, elements of number theory, modular arithmetic, and finite fields. Shaping the algebraic curriculum should go in step with development of the bespoke, and very child-friendly, Domain-Specific Language (DSL)[2] for use in the course. Most likely, existing general purpose languages are not suitable for this role. In particular, DSL is needed for bridging the conceptual gap between

- formulation of an algorithm and its implementation in a code, and
- representation of the answer as a closed algebraic formula.

1.3. Elementary Geometry

AAA includes elementary geometry as a theory of the 2-dimensional vector space over $\mathbb{R}$ with two non-degenerate bilinear forms, one symmetric—*dot product*, another skew-symmetric—*wedge product*. This approach was developed by Israel Gelfand in his proto-textbook of elementary geometry for schools, left, unfortunately, unfinished. The use of the wedge product allows us to easily handle, at the level of algorithms and computer code, orientation of the plane and concepts of left/right and clockwise/counterclockwise. Moreover, this solves problems unapproachable in the traditional school geometry, for example, this one:

> **Problem.** A scalene triangle (that is, a triangle in which all three sides are in different lengths) is drawn on a piece of paper. It was carefully cut out and turned over. Prove that if you move this triangle along a sheet of paper, then it cannot be used to cover the hole left behind without any gaps.

You can experiment with a cut out and flipped over scalene triangle on a Möbius strip; if you drag it near the hole, you can't cover the hole. But if you drag it along the entire length of the strip, it covers it perfectly. Locally, a Möbius strip is a piece of the Euclidean plane; but, unlike the latter, it is not orientable [8].

[2] A Domain-Specific Language (DSL) is a computer language specialized to a particular application domain. This is in contrast to a general-purpose language (GPL), which is broadly applicable across domains. Famous examples include html and LaTeX. Some books: [2, 11].

Orientability of a surface is its *global* property, but school geometry was always *local*, made on a small sheet of paper.

1.4. The rest of school mathematics

Other parts of mathematics, for example, mechanics, combinatorics, probability theory, and statistics need a separate discussion and are not touched here. Also, interactions with physics deserve most serious attention.

Conclusion

This abstract is only a very brief outline of a relatively short about talk proposed course. Some additional details could be found in [7].

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Basic representations of affine Lie algebras and theory of Chevalley groups

Igor Frenkel

Basic representation of affine Lie algebras when restricted to finite dimensional simple subalgebra gives a model of all irreducible finite dimensional representations. Thanks to its elementary nature the basic representation is very useful in various classical problems of simple Lie algebras and Chevalley groups (and we discussed this with Kolya). We recall a construction of the basic representation and present some new developments of this construction that have a potential of further applications to classical theory.

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Some problems around $SL_2(C)$

Nikolai Gordeev

Let G be a group and let F_n be a free group of the rank n. For a word $w \in F_n$ the word map $\widetilde{w} : G_n \to G$ is defined by the formula $\widetilde{w}((g_1, \ldots, g_n)) = w(g_1, \ldots, g_n)$. During the last 10-15 years the theory of word maps are intensively developed especially for the case when G is a simple algebraic group. Here we consider some problems of the theory of word maps and related questions for the case when $G = SL_2(C)$

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Tropical vs. classical algebra

Dima Grigoriev

Tropical (or min-plus) algebra have numerous common features with classical commutative algebra, while its statements and proofs considerably differ from their classical counterparts. We consider tropical analogs of linear algebra, Nullstellensatz, Milnor-Thom inequality, Hilbert polynomial.

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Matrix evaluations of non-commutative polynomials

Alexey Kanel-Belov, L.Rowen, Sergey Malev

The talk is dedicated to the memory of Nikolai Vavilov. We extend his ideas on group word mappings to the ring case. It is joint work with Sergey Malev and Louis Rowen.

The general statement is as follows. Let $P(x1, \ldots, x_n)$ be a non-commutative polynomial in matrices of order n. What can be said about the set of its values?

I. Kaplansky and I. V. Lvov posed the question (see Dniester Notebook) that the set of values of a multilinear polynomial is a vector space (in this case it coincides either with zero, or with the space of all matrices, or with the space of traceless matrices, or with scalar matrices). The solution of Kaplansky's problem even for second-order matrices over a square-closed field turned out to be quite non-trivial and deep. It got many references. Recently it was solved for Calley algebra.

Questions related to equations in matrices, in addition to their applied significance, are related to the construction of an algebraically closed field, to the freedom theorem: if we add a new non-commutative variable and a relation where it is involved, this will not lead to the appearance of new relations. There are a number of deep problems related to the set of values of words in a group, in particular in second-order matrices.

We discourse state of art of this problem, some partial results, Lie algebra case and some relations with group theory.

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Finite groups without elements of order $2p$ for an odd prime p

A.S. Kondratiev, J. Guo, W. Guo and M.S. Nirova

Abstract. We consider the problem of desription of finite groups without elements of order $2p$ for an odd prime p. A rich history of the study of particular cases of the problem is given. As main result, we prove new general structural theorem on finite non-solvable groups without elements of order $2p$ for an odd prime p. The theorem reduces largely a solving the problem to investigating almost simple groups and faithful 2-modular representations of quasisimple-groups. The theorem reinforces essentially well-known Vasil'ev-Gorshkov theorem and can be applied to obtaining new arithmetical characterizations of finite groups.

Our notation and terminology are mostly standard and can be found in [1, 2].

Let G be a finite group. Denote by $\pi(G)$ the set of all prime divisors of the order of G. The Gruenberg–Kegel graph, or the prime graph, of G is the graph $\Gamma(G)$ with vertex set $\pi(G)$ in which two vertices p and q are adjacent if and only if $p \neq q$ and G has an element of order pq. A set of a pairwise non-adjacent vertices of a graph is called its coclique. Denote by $t(G)$ the greatest cardinality of cocliques of $\Gamma(G)$ and by $t(2, G)$ the greatest cardinality of cocliques containing 2 of $\Gamma(G)$. A finite group X is almost simple whenever $S \leq X \leq Aut(S)$ for some finite nonabelian simple group S; equivalently, provided that the socle of X is a finite nonabelian simple group. A finite group X is quasisimple whenever $X' = X$ and $X/Z(X)$ is a finite nonabelian simple group.

It is well known the dominating role of involutions in the thery of finite non-solvable groups and especially in the classification of finite simple groups (CFSG). If G is a finite group of even order and $2 \neq p \in \pi(G)$ then it is important to know whether the vertices 2 and p are adjacent in the graph $\Gamma(G)$. We shall call an W_p-group for an odd prime p a finite group of order dividing by $2p$ and having no elements of order $2p$; the vertices 2 and p in the Gruenberg–Kegel graph of such group are non-adjacent. Denote by W the class of all W_p-groups when p runs all odd primes. Then W coincides with the class of finite groups G such that $t(2, G) \geq 2$.

Any finite simple group belongs to the class W, except the alternating group of some degree. Finite groups with disconnected Gruenberg-Kegel graphs belong also to W. Thus, the class W is wide and the following problem arises naturally.

Problem. *Describe W_p-groups for an odd prime p, at least small ones.*

Given a finite group G and an odd prime p, denote by $W_p(G)$ the group $O^{\{2,p\}'}(G/O_{\{2,p\}'}(G))$ It is clear that G is a W_p-group if and only if $W_p(G)$ is a W_p-group.

The history of results on particular cases of Problem is very rich.

Burnside (1900) considered the case when the order of any element of a finite group either is odd or equals 2.

G. Higman (1957) described finite groups whose element orders are prime powers (Shi called such groups shortly as $EPPO$-groups). It is clear that the connected components of the Gruenberg-Kegel graph of a $EPPO$-group are one-element.

A finite group of order dividing by a prime p is called a C_{pp}-group if the centralizer of any its non-trivial element is a p-group. It is clear that a $EPPO$-group G is C_{pp}-group for any $p \in \pi(G)$.

Back at the dawn of classification finite simple groups (CFSG), M. Suzuki (1961-1962) in his pioneer fundamental papers obtained a description of C_{22}-groups, i. e., groups which are W_p-groups for all odd prime divisors p of their orders. In the sequal, this description was refined and reduced to a criterion in works of G. Higman (1968), P. Martino (1972), Stewart (1973) and Brandl (1981). As a corollary, a complete description of $EPPO$-groups was obtained.

A complete description of non-primary C_{33}-groups is obtain by G. Higman (1968), Stewart (1973) and Fletcher, Stellmacher, and Stewart (1977).

A description of non-primary C_{55}-groups is obtained in the papers by Dolphi, Jabara, Lucido (2004) and Astill, C. Parker and Waldecker (2012).

It is remained open the problem of describing C_{pp}-groups for a prime $p > 5$.

It is clear that the Gruenberg–Kegel graph of any non-primary C_{pp}-group is disconnected.

The first result about finite groups with disconnected Gruenberg-Kegel graph is the well-known structural theorem (Gruenberg-Kegel Theorem) obtained by Gruenberg and Kegel about 1975 in an unpublished paper. The proof of this theorem was published in the paper of Williams (1981), post-graduate of Gruenberg.

Williams (1981) obtained also an explicit description of connected components of the Gruenberg-Kegel graph for all finite simple non-abelian groups except the groups of Lie type of even characteristic.

In 1989, AK [4] obtained such description for the remaining case of the groups of Lie type of even characteristic. Later in 1993, this result was repeated by Iiyori and Yamaki. But later, some inaccuracies in all three papers were found. In a joint work of AK and Mazurov [5], the corresponding tables were corrected.

The classification of connected components of Gruenberg-Kegel graph for finite simple groups were applied by Lucido (1999, 2002) for obtaining analogous classification for all finite almost simple groups.

It is remained open the natural problem of describing the finite non-solvable groups with disconnected Gruenberg-Kegel graph, which are not almost simple.

In 2005, A.V. Vasil'ev observed that the proof of the Gruenberg-Kegel Theorem uses essentially the fact that a finite group G with disconnected Gruenberg–Kegel graph contains an element of odd prime order which does not adjanced in Γ to 2. A.V. Vasil'ev in [7] proved a wide generalization of the Gruenberg-Kegel Theorem for non-solvable groups. This result was a few sharpened by A.V. Vasil'ev and Gorshkov in [8]. Vasil'ev-Gorshkov Theorem can be considered as a general structural theorem for non-solvable groups from class W with an emphasis on a relation with their Gruenberg-–Kegel graphs.

Now we consider our results on Problem.

The non-abelian finite simple W_3-groups were determined in 1977 in the three independent articles of Podufalov, Fletcher, Stellmacher, and Stewart, as well as Gordon. The problem of describing general finite W_3-groups remained open for more than 40 years before In 2018, AK and Minigulov in [6] solved it without using CFSG.

Recently in [3], we solved Problem for solvable W_p-groups and $p > 3$. Now using this result, we reinforce Vasil'ev-Gorshkov Theorem with an emphasis on normal structure of investigated groups proving the following main theorem.

Theorem. *Let G be a non-solvable W_p-group for an odd prime p, $K = S(G)$ and $\overline{G} = G/K$. Then $\overline{G}$ is an almost simple group with the socle S, p does not divide the index $|\overline{G} : S|$, $t(S) \geq t(G)-1$, $t(2,S) \geq t(2,G)$, and one of the following assertions holds:*

(1) *$K = O_{\{2,p\}'}(G))$;*

(2) *p divides $|K|$, a Sylow 2-subgroup of G is (generalized) quaternion group, $G/O(G)$ is isomorphic to either $2{\cdot}A_7$ or an extension of $SL_2(q)$ for odd $q > 3$ by a cyclic group of either odd order or a doubled odd order, $O_{p',p,p'}(O(G)) = O(G)$, an involution from $K = Z^*(G)$ inverts some (abelian) Sylow p-subgroup of $O(G)$ and centralizes $O(G)/O_{p',p}(O(G))$, $t(G) = t(2,S) = 3$, and $t(2,G) = 2$;*

(3) *p does not divide $|K|$, 2 divides $|K|$, a Sylow p-subgroup of G is cyclic, $O_{2',2,2'}(K) = K$ and S centralizes $K/O_{2',2}(K)$.*

Remark. In the case (3) of Theorem, the last term E of the derived series of $G/O_{2',2}(K)$ is a quasisimple W_p-group such that $EK/O_{2',2}(K)$ is a central product of $K/O_{2',2}(K)$ and E, $E/Z(E)$ is isonorphic to S, and E acts faithfully on $O_{2',2}(K)/O(K)$.

Therefore, Theorem reduces largely a solving Problem to investigating the cases (1) and (3) of Theorem, i. e., almost simple W_p-groups and faithful 2-modular representations of quasisimple W_p-groups.

Recently, we determined in [3] all almost simple W_5-groups.

The results we have obtained can be applied to obtaining new arithmetical characterizations of finite groups.

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On unitary Nil K_1-groups

Viacheslav Kopeiko

Abstract. In the paper we introduce a new unitary nil subgroup of the unitary Bass' nilpotent K_1-group of a unitary ring. This group is a extending of a unitary nil K_1-group is defined before by author. The properties of the nil group are unitary analogues well-known properties of the Bass' nilpotent K_1-group of a ring in algebraic K-theory.

Introduction

In the paper, we follow the standard setting and notations of unitary (algebraic) K-theory [1]. Let (R, λ, Λ) be a unitary ring, where R is an associative ring with 1, equipped with an involution $x \to \bar{x}$. Further, let λ be a central element of R such that $\lambda \cdot \bar{\lambda} = 1$, and let Λ be an additive subgroup of R such that $\{x - \lambda\bar{x}, x \in R\} \leq \Lambda \leq \{x \in R : x = -\lambda\bar{x}\}$. We note that $(R, \bar{\lambda}, \overline{\Lambda})$, where $\overline{\Lambda} = \{\bar{x}, x \in \Lambda\}$, is a unitary ring also. Let us extend the involution to the matrix ring $M_r(R)$ by setting $(a_{ij})^* = (\overline{a_{ji}})$.

Definition 1. A matrix $a (\in M_r(R))$ is said to be Λ-hermitian if the a is a $(-\lambda)$-hermitian, i.e., $a = -\lambda a^*$, and all diagonal entries of the a belong to Λ.

For a natural number r we set $I_r^\lambda = \begin{pmatrix} 0 & e_r \\ \lambda e_r & 0 \end{pmatrix}$, where e_r is the identity matrix of degree r.

Definition 2. A matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2r}(R)$, where $a, b, c, d \in M_r(R)$, is said to be unitary if $\alpha^* I_r^\lambda \alpha = I_r^\lambda$; and the α to be Λ-unitary if moreover the diagonal entries of the matrices a^*c and b^*d are contained in Λ.

The set $U_{2r}^\lambda(R, \Lambda)$ of all Λ-unitary matrices of degree $2r$ forms a group; it is called the (hyperbolic) Λ-unitary group. Denote by $EU_{2r}^\lambda(R, \Lambda)$ the subgroup of $U_{2r}^\lambda(R, \Lambda)$, generated by all matrices of the form $H(a) = \begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1} \end{pmatrix}$ (hyperbolic matrix); $\begin{pmatrix} e_r & b \\ 0 & e_r \end{pmatrix}$; $\begin{pmatrix} e_r & 0 \\ c & e_r \end{pmatrix}$, where $a \in E_r(R)$, b is a $\overline{\Lambda}$-hermitian, and c is a Λ-hermitian. The group $EU^{\lambda}_{2r}(R,\Lambda)$ is called elementary (hyperbolic) Λ-unitary group.

Let us define an embedding $U^{\lambda}_{2r}(R,\Lambda) \to U^{\lambda}_{2r+2}(R,\Lambda)$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and set $U^{\lambda}(R,\Lambda) = \cup U^{\lambda}_{2r}(R,\Lambda)$, $EU^{\lambda}(R,\Lambda) = \cup EU^{\lambda}_{2r}(R,\Lambda)$.

In view of the unitary analog of the Whitehead lemma ([1], Chap.2, Proposition 3.7), $EU^{\lambda}(R,\Lambda) = [U^{\lambda}(R,\Lambda), U^{\lambda}(R,\Lambda)]$. In particular, the (abelian) group $K_1U^{\lambda}(R,\Lambda) = U^{\lambda}(R,\Lambda)/EU^{\lambda}(R,\Lambda)$ is well defined. The class of a matrix $\alpha \in U^{\lambda}_{2r}(R,\Lambda)$ in the group $K_1U^{\lambda}(R,\Lambda)$ is denoted by $[\alpha]$. As a result, we obtain unitary K_1-functor K_1U acting from the category of unitary rings to the category of abelian groups.

Definition 3. The kernel of the (splitting) group epimorphism $K_1U^{\lambda}(R[X],\Lambda[X]) \to K_1U^{\lambda}(R,\Lambda)$ induced by unitary surjection of unitary rings $(R[X],\Lambda[X]) \to (R,\Lambda) : X \to 0$ is called unitary Bass' nilpotent K_1-group of the unitary ring R and it is denoted by $NK_1U^{\lambda}(R,\Lambda)$.

Definition 4. A nonzero matrix $\alpha = \begin{pmatrix} a & -b \\ c & -a^* \end{pmatrix}$ is called the second order nilpotent of the Λ-unitary type if its components $a, b, c \in M_r(R)$ satisfies the following conditions:

1) the matrices b and ab are $\bar{\Lambda}$-hermitian, and $ab = ba^*$;

2) the matrices c and ca are Λ-hermitian, and $ca = a^*c$;

3) $bc = a^2$ and $cb = (a^*)^2$.

It is easy to check that α is a nilpotent matrix of nilpotency degree 2.

In the sequel we abbreviate of I^{λ}_r to I.

Proposition 1. The matrix $\alpha(\in M_{2r}(R))$ is a second order nilpotent of the Λ-unitary type if and only if $\alpha^*I + I\alpha = 0$ and $\alpha^*I\alpha = 0$.

The following result was proved by author in [2].

Theorem 1. Let α be a nonzero matrix in $M_{2r}(R)$. Then for any natural number m the matrix $e_{2r} - \alpha X^m$ is a $\Lambda[X]$-unitary matrix if and only if the matrix α is a second order nilpotent of the Λ-unitary type.

Recall definition of a unitary nil group is introduced by author in [2]. We denote by $Unip_1K_1U^{\lambda}(R,\Lambda)$ the subgroup of $NK_1U^{\lambda}(R,\Lambda)$ generated by all elements of the type $[e_{2r} - \alpha X^m]$, where r, m are natural numbers, and $\alpha(\in M_{2r}(R))$ is a second order nilpotent of the Λ-unitary type.

Let us consider a number of examples.

Example 1. Let $e_{2r} - \alpha X$ be a $\Lambda[X]$-unitary matrix, where $\alpha = \alpha(X) = a_1 + a_2X$, and a_1, a_2 are nonzero matrices over R. Suppose $\alpha^2 = 0$. Then $a_1^2 = 0$, $a_2^2 = 0$, and $a_1a_2 + a_2a_1 = 0$. From these equalities it follow that $(a_1a_2)^2 = 0$. Further, by definition, $(e_{2r} - \alpha X)^*I(e_{2r} - \alpha X) = I$. Therefore we obtain that $a_1^*I + Ia_1 = 0$, $a_2^*I + Ia_2 + a_1^*Ia_1 = 0$, $a_1^*Ia_2 + a_2^*Ia_1 = 0$, $a_2^*Ia_2 = 0$. Then from the equalities $a_1^2 = 0$ and $a_1^* I + I a_1 = 0$ we get $a_1^* I a_1 = 0$. Hence, $a_2^* I + I a_2 = 0$. Moreover from the equalities $a_1^* I + I a_1 = 0$ and $a_1^* I a_2 + a_2^* I a_1 = 0$ it follow that $(a_1 a_2)^* I + I(a_1 a_2) = 0$. Hence a_1, a_2, and $a_1 a_2$ are a second order nilpotent of the Λ-unitary type by Proposition 1, and we obtain the following decomposition of the unitary unipotent matrix into the product of the unitary unipotent matrices $e_{2r} - a_1 X - a_2 X^2 = (e_{2r} - a_1 X)(e_{2r} - a_2 X^2)(e_{2r} - a_1 a_2 X^3)$. Thus $[e_{2r} - \alpha X] = [e_{2r} - a_1 X][e_{2r} - a_2 X^2][e_{2r} - a_1 a_2 X^3] \in Unip_1 K_1 U^\lambda(R, \Lambda)$.

Now we formulate a more general statement.

Proposition 2. Let $e_{2r} - \alpha X$ be a $\Lambda[X]$-unitary matrix, where $\alpha = \alpha(X) = a_1 + a_2 X + \ldots + a_n X^{n-1}$, $a_i \in M_{2r}(R)$. Suppose $\alpha^2 = 0$ and $a_i a_j + a_j a_i = 0$ for all $1 \leq i, j \leq n$. Then all nonzero matrices a_i and $a_i a_j$ are a second order nilpotent of the Λ-unitary type, and we have decomposition $e_{2r} - X\alpha = \prod_{i=1}^n (e_{2r} - a_i X^i) \prod_{1 \leq i < j \leq n}^n (e_{2r} - a_i a_j X^{i+j})$. Hence, $[e_{2r} - X\alpha] \in Unip_1 K_1 U^\lambda(R, \Lambda)$.

The following example shows necessity the assumption $a_i a_j + a_j a_i = 0$ for all $1 \leq i, j \leq n$ in Proposition 2.

Example 2. Let $e_{2r} - \alpha X$ be a $\Lambda[X]$-unitary matrix, where $\alpha = \alpha(X) = a_1 + a_2 X + a_3 X^2$, a_1, a_2, and a_3 are nonzero matrices over R. Suppose $\alpha^2 = 0$. By analogy with investigation in Example 1 we get that a_1 and a_3 are a second order nilpotent of the Λ-unitary type, $a_1 a_2 + a_2 a_1 = 0$, $a_1 a_3 + a_3 a_1 + a_2^2 = 0$, and $a_2 a_3 + a_3 a_2 = 0$. If $a_1 a_3 + a_3 a_1 = 0$, then $a_2^2 = 0$. Hence in this case, by Proposition 2, we obtain decomposition $[e_{2r} - \alpha X] = [e_{2r} - a_1 X][e_{2r} - a_2 X^2][e_{2r} - a_3 X^3][e_{2r} - a_1 a_2 X^3][e_{2r} - a_1 a_3 X^4][e_{2r} - a_2 a_3 X^5] \in Unip_1 K_1 U^\lambda(R, \Lambda)$. In the other side, i.e., if $a_1 a_3 + a_3 a_1 \neq 0$, we have only decomposition $[e_{2r} - \alpha X] = [e_{2r} - a_1 X][e_{2r} - a_2 X^2 - a_1 a_2 X^3 - a_1 a_3 X^4 - a_2 a_3 X^5 - a_1 a_2 a_3 X^6][e_{2r} - a_3 X^3]$. I don't know, but I conjecture the element $[e_{2r} - a_2 X^2 - a_1 a_2 X^3 - a_1 a_3 X^4 - a_2 a_3 X^5 - a_1 a_2 a_3 X^6]$ is not belongs in $Unip_1 K_1 U^\lambda(R, \Lambda)$. Hence the nil group $Unip_1 K_1 U^\lambda(R, \Lambda)$ must be extended.

We denote by $\overline{Unip_1} K_1 U^\lambda(R, \Lambda)$ the nil subgroup of $NK_1 U^\lambda(R, \Lambda)$ generated by all elements of the type $[e_{2r} - X\alpha]$, where r is a natural number, $\alpha = \alpha(X) (\in M_{2r}(R[X]))$ is a nilpotent matrix of nilpotency degree 2. By definition, $Unip_1 K_1 U^\lambda(R, \Lambda)$ is a subgroup of $\overline{Unip_1} K_1 U^\lambda(R, \Lambda)$.

Theorem 2. 1) Let n be a positive integer such that $n = n \cdot 1 = 0$ in the ring R, where 1 is the identity element of R. Then the group $\overline{Unip_1} K_1 U^\lambda(R, \Lambda)$ is of an n-torsion group.

2) If n is an invertible element of R, then the group $\overline{Unip_1} K_1 U^\lambda(R, \Lambda)$ is a uniquely divisible by n.

3) If R is a $\mathbf{Q}$−algebra, where $\mathbf{Q}$ denotes the field of rational numbers, then $\overline{Unip_1} K_1 U^\lambda(R, \Lambda)$ is a $\mathbf{Q}$−vector space. In particular, $\overline{Unip_1} K_1 U^\lambda(R, \Lambda)$ is a divisible group, and it is a direct summand of the group $NK_1 U^\lambda(R, \Lambda)$.

In [2] we is also introduced the nil subgroup $Unip_2 K_1 U^\lambda(R, \Lambda)$ of $NK_1 U^\lambda(R, \Lambda)$, generated by all elements of the type $[H(e_r - aX)]$, where $a (\in M_r(R))$ is a nilpotent matrix. For the nil groups $Unip_1 K_1 U^\lambda(R, \Lambda)$, $Unip_2 K_1 U^\lambda(R, \Lambda)$ analogous of properties 1)-3) from Theorem 2 is proved in [2]. But in the case of the group

$Unip_2K_1U^\lambda(R,\Lambda)$ in 2) and 3) we must propose that either hyperbolic homomorphism $H : K_1(R) \longrightarrow K_1U^\lambda(R,\Lambda) : [a] \longrightarrow [H(a)]$ or the forgetful homomorphism $F : K_1U^\lambda(R,\Lambda) \longrightarrow K_1(R) : \alpha\ modEU^\lambda(R,\Lambda) \longrightarrow \alpha\ modE(R)$ is a group monomorphism.

Conclusion

In this paper we introduce a new type of the unitary nil group $\overline{Unip_1}K_1U^\lambda(R,\Lambda)$ of a unitary ring, and we investigate some properties of this nil group. The group $\overline{Unip_1}K_1U^\lambda(R,\Lambda)$ is a extending of the unitary nil group $Unip_1K_1U^\lambda(R,\Lambda)$ is defined before by author. Its properties are unitary analogues well-known properties of the Bass' nilpotent K_1-group of a ring in algebraic K-theory [3].

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Algebras of permutations

Sergei Lando

The group algebras $C[S_m]$ of the symmetric groups and, especially, their centers $ZC[S_m]$ play a crucial role in the study of the structure of these groups and their representations. Moreover, they are widely used in the analysis of geometry of moduli spaces of algebraic curves. In particular, Hurwitz numbers,

which enumerate ramified coverings of the projective line with prescribed ramification types, can be interpreted as connection coefficients of certain classes

generating $ZC[S_m]$ (A. Hurwitz, 1891). This understanding led to a relationship between Hurwitz numbers and integrable systems of mathematical physics (A. Okounkov, 2000). Recently, in the study of real ramified coverings of the projective line, different versions of these algebras, which are called algebras of transition types, naturally appeared (M. Kazarian, S. Lando, S. Natanzon, 2021),. Other recent appearances of algebras associated to permutations are related to weight systems, which are a combinatorial counterpart of finite type invariants of knots. Originally, weight systems were defined as functions on chord diagrams, which can be understood as permutations of special kind, namely, involutions without fixed points (V. Vassiliev, 1990). Last years, an idea due to M. Kazarian allowed for extending them to arbitrary permutations and to developing efficient algorithms for computing weight systems (M. Kazarian, S. Lando, Z. Yang). The talk will be devoted to describing these objects and to discussing various possible relationships between them.

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Uniform stability of high-rank Arithmetic groups

Alex Lubotzky

Lattices in high-rank semisimple groups enjoy several special properties like super- rigidity, quasi-isometric rigidity, first-order rigidity, and more. In this talk, we will add another one: uniform (a.k.a. Ulam) stability. Namely, it will be shown that (most) such lattices D satisfy: every finite-dimensional unitary "almost-representation" of D (almost w.r.t. to a sub-multiplicative norm on the complex matrices) is a small deformation of a true unitary representation. This extends a result of Kazhdan (1982) for amenable groups and Burger-Ozawa-Thom (2013) for $SL(n, Z), n > 2$. The main technical tool is a new cohomology theory ("asymptotic cohomology") related to bounded cohomology similar to the connection of the last one with ordinary cohomology. The vanishing of H^2 w.r.t. to a suitable module implies the above stability.

The talk is based on a joint work with L. Glebsky, N. Monod, and B. Rangarajan.

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Algebraic groups: abstract isomorphisms, Diophantine problems, and model theory

Alexei Miasnikov

Abstract. In this talk I will mostly focus on three problems: describing abstract automorphisms of a given algebraic group G; describing which algebraic groups are elementarily equivalent to G; and describing which abstract groups are elementarily equivalent to G. Recent results on the classical matrix groups, Chevalley groups, and unipotent groups indicate that there are some powerful uniform methods to approach all these problems. These methods are based on the technique of interpretations from model theory. I will also show some applications of these methods to Diophantine problems in algebraic groups.

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On the Grothendieck–Serre conjecture

Ivan Panin

A recent view point on the conjecture will be provided. This will allow to explain to participants as known results so the current stage of the art. Recall that the conjecture states that rationally trivial G- bundle are Zariski locally trivial, provided that G is reductive and the base of the G- bundle is regular (or at least smooth).

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Division algebras and algebraic groups: structure and cohomology

Andrei Rapinchuk

Abstract. The talk is a tribute to the Saint Petersburg algebraic school. We will begin with Faddeev's theorem on the Brauer group of the field of rational functions, and discuss the extension of its most essential part to arbitrary reductive algebraic groups proposed by Raghunathan and Ramanathan. We will sketch a recently found short proof of the Raghunathan-Ramanathan theorem that uses building-theoretic techniques. These techniques are also instrumental in the analysis of the structure of groups of points of algebraic groups - the area to which Nikolay Vavilov has made major contributions. We will first describe finite subgroups of the groups of points of reductive algebraic groups over polynomial rings of characteristic zero, and then turn to the problem of bounded generation of Chevalley groups over rings of arithmetic type by root subgroups. Here we will emphasize the case where the base ring is the coordinate ring of a geometrically integral smooth curve over a finite field, and survey the results of Kunyavskii-Plotkin-Vavilov. In the concluding part of the talk, we will introduce the genus of a simple algebraic group, and highlight one application of the Raghunathan-Ramanathan theorem to the genus problem. We will close with the notion of motivic genus proposed by Merkurjev and the relevant results of Izboldin, Karpenko and Vishik.

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N. Vavilov and Digital Foundations of Mathematics and Mathematical Education

Alexei Semenov

Abstract. The talk addresses activities of the outstanding mathematician Nikolai Alexandrovich Vavilov and the results of this activity, which lie outside the theorems proved by him and the definitions proposed by him.

I will talk about three areas that are connected and intersect in Vavilov's world:

- Computer in mathematical research
- Philosophy of mathematics and foundations of mathematics
- The educational meaning and significance of mathematics

In addition to personal communication with N.A., I rely primarily on his works, a list of which is given at the end of this abstract.

In particular, among them is a review of the applications of the computer for research in number theory [1, 2, 3, 4, 5, 6]. The total number of references read by Nikolai in this review is 1765.

Note that Vavilov's views on the use of computers in mathematics and mathematical education are based on technologies mainly formed in the 1980s (or early 1990s). I consider this as a positive factor in our discussion. The first technology is a mathematical text editor designed by one person — Donald Knuth; this editor has fundamentally reduced the distance between the mathematician who wrote a text and the mathematician who reads it. The second are computer algebra systems that have reduced the distance between the research mathematician and the mathematical reality under observation and study. Vavilov mentions and used various CAS, but Mathematica was still the main resource for him in the educational field. It was also designed by one man, Steven Wolfram. In a sense, considering this stage of technology is important to us today.

We have to understand clearly what can be done successfully, BEFORE the advent of big data and big players in the field. As you know the results of the advent are being published in Nature with 5–10 authors from a leading corporation and a top university.

In the talk, I will try to analyze the oppositions and relationships formulated by Vavilov based on decades of rational AI preceding the years of intuitive AI and creative AI. Among them:

- An ideal mathematical reality and an experiment in it
- Coordination of ideas and calculations, theory and experiment
- Deterministic and random
- The simplicity of a mechanism for generation and the unpredictable complexity of the generated
- Various finitnesses, including virtually infinite and intermediate ones
- Proof and error in mathematics
- etc.

In the talk, I will present my concept of the complexity of objects, which is relevant to the discussion about different finitnesses. I propose to consider the complexity of the finite object relative to a mode (program) of description as the sum of the length of the mode and the length of the shortest description. With this definition, the optimal way of describing will be worse by only additive dozens, or at least hundreds, than other, as Kolmogorov believed. So, when talking about 'not-so-real' numbers I propose to consider numbers not being big (Knuth a.o.) but having high complexity

Many fundamental positions related to the philosophy of mathematics and mathematical activity, the role of evidence, proof, and errors in this activity are reflected in [7]. In connection with the discussion of our ideas about mathematical proofs, I propose an interpretation of Hilbert's program as a goal to prove that there is no contradiction in mathematics with complexity less than the size of the Universe.

Another fundamental text by Vavilov [8] refers to his educational principles. I forgot to ask Nikolai Alexandrovich about the meaning of the initial metaphor of the title. I hope his co-authors will explain this to me at the conference. My, external reconstruction of meaning: "Don't panic!" Vavilov says that "Teaching mathematics should intrigue, captivate and fascinate". I will try to point out two features of learning aimed at implementing this principle for all students: solving problems that are not known how to solve, and the ability to use digital technologies.

The speaker hopes for the active participation of listeners, in particular colleagues and co-authors of Nikolai Vavilov in his educational and other endeavors. Such participation will make it possible to more fully restore, preserve and develop Nikolai Vavilov's legacy in the areas.

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Some formulas for calculating structure constants for Lie algebras of types B_n and C_n

Rafael Stekolshchik

Abstract. In [V04], Vavilov observed that in the Carter formula for calculating structure constants in the simply-laced case, only two of three summands can be nonzero. This observation motivated the question of how things stand in the multiply-laced case. To answer this question the concepts of *special pair* and *extraspecial pair* introduced by Carter [Ca72] are used. Let $\{r, s\}$ be a special pair of roots for which the structure constant $N(r, s)$ is sought, and let $\{r_1, s_1\}$ be the extraspecial pair of roots corresponding to $\{r, s\}$. The concept of the *Carter quartet* $\{r_1, r, s, s_1\}$ is introduced. Carter quartets are the main object of interest. Several relations are given in terms of Carter quartets for the Lie algebras of types B_n and C_n. These relations make it possible to calculate the structure constants $N(r, s)$ in B_n using the formula of the simply-laced case, and in C_n - using the same formula, but up to a certain parameter, which depends only on the extraspecial pair of roots $\{r_1, s_1\}$.

1. Formulas of Chevalley, Tits and Carter

The structure constant $N(\alpha, \beta)$ defines the value of the Lie bracket on the root vectors e_α and e_β in the simple Lie algebra $\mathfrak{g}$:

$$[e_\alpha, e_\beta] = \begin{cases} N(\alpha,\beta)e_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi, \\ 0 & \text{if } \alpha+\beta \notin \Phi. \end{cases}$$

where Φ is a root system associated with $\mathfrak{g}$. In 1955, Chevalley [Ch55] showed how the structure constants of the simple Lie algebra $\mathfrak{g}$ can be calculated.

Theorem (Chevalley, 1955). *The root vectors e_α can be chosen so that*

$$[e_\alpha, e_{-\alpha}] = h_\alpha \text{ for all } \alpha \in \Phi,$$
$$N(\alpha, \beta) = -N(-\alpha, -\beta) \text{ for all } \alpha, \beta \in \Phi.$$

Let $\alpha, \beta \in \Phi$ such that $\alpha + \beta \in \Phi$ and p be the greatest integer such that $\beta - p\alpha \in \Phi$. Then one has

$$N(\alpha, \beta) = \pm(p + 1), \tag{1.1}$$

see [Se01, Ch. 6,§6], [Ch55, Th. 1].

The absolute value of the structure constant $N(\alpha, \beta)$ in (1.1) is determined only by the length of α-series of roots through β. There is just one catch left, how to determine the signs of the structure constants. Further advances in calculations of the structure constants were largely due to the works of Tits [T66] and Carter [Ca72].

Theorem (Carter, 1972). *The structure constants of a simple Lie algebra* $\mathfrak{g}$ *over* $\mathbb{C}$ *satisfy the following relations:*

$$N(\alpha,\beta) = -N(\beta,\alpha), \qquad \alpha,\beta \in \Phi. \tag{1.2}$$

$$\frac{N(\alpha,\beta)}{|\gamma|^2} = \frac{N(\beta,\gamma)}{|\alpha|^2} = \frac{N(\gamma,\alpha)}{|\beta|^2} \tag{1.3}$$

if $\alpha,\beta,\gamma \in \Phi$ *and* $\alpha+\beta+\gamma = 0$.

$$\frac{N(\alpha,\beta)N(\gamma,\delta)}{|\alpha+\beta|^2} + \frac{N(\beta,\gamma)N(\alpha,\delta)}{|\beta+\gamma|^2} + \frac{N(\gamma,\alpha)N(\beta,\delta)}{|\alpha+\gamma|^2} = 0 \tag{1.4}$$

if $\alpha,\beta,\gamma,\delta \in \Phi$, $\alpha+\beta+\gamma+\delta = 0$ *and if no pair are opposite. If one of sums in* (1.4) *is not a root, the corresponding structure constant is 0, and the corresponding term in* (1.4) *disappears.*

Relation (1.3) is due to J. Tits, [T66]. Relation (1.4) is due to R. Carter. This relation is the basic formula for calculation of the structure constants.

2. Vavilov's observation for simply-laced case

For the simply-laced case, the length of any root is 1, and Carter's formula (1.4) looks significantly simpler:

$$\begin{gathered} N(\alpha,\beta)N(\gamma,\delta) + N(\beta,\gamma)N(\alpha,\delta) + N(\gamma,\alpha)N(\beta,\delta) = 0, \\ \text{if } \alpha+\beta+\gamma+\delta = 0. \end{gathered} \tag{2.1}$$

For details on how to find the signs of the structure constants $N(\alpha,\beta)$, see [V04], [C03]. In [V04], Vavilov uses Kac-Frenkel cocycles to compute the cases E_6, E_7 and E_8, and this algorithm is significantly faster than inductive algorithm proposed by Tits, [T66]. However, here, I would like to highlight one observation made by Vavilov. He noticed that only two summands in eq. (2.1) can be non-zero. The proof is as follows: if $N(\alpha,\beta) \neq 0$ and $N(\beta,\gamma) \neq 0$ then $\alpha+\beta$ and $\beta+\gamma$ both are roots, and angles $\widehat{\beta,\alpha}$ and $\widehat{\beta,\gamma}$ both are $2\pi/3$. Suppose $N(\alpha,\gamma) \neq 0$, then the angle $\widehat{\alpha,\gamma}$ also equals $2\pi/3$. Then α,β,γ lie in one plane and $\alpha = -(\beta+\gamma)$, see Fig. 1. Then, according to (2.1) $\delta = 0$, a contradiction.

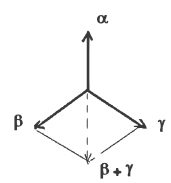

FIGURE 1. The angle between each pair of roots from the triple $\{\alpha,\beta,\gamma\}$ is $2\pi/3$.

This observation motivated me to check how things are in the multiply-laced case. In this end, the so-called extraspecial pairs of roots introduced by Carter are needed.

3. Extraspecial pairs, Carter's formula and Carter quartets

The positive roots in any root system can be ordered in such a way that a root of smaller height precedes a root of larger height, and the roots of the same height are ordered according to the *lexicographic* rule. The symbol $\prec$ is used to denote the relation "precede". This ordering of positive roots is called *regular*, [V04, p. 67].

An ordered pair $\{r, s\}$ of positive roots is called a *special pair* if $r + s \in \Phi$ and $0 \prec r \prec s$. Here, the special pairs are defined only for the positive roots. For the transition from the positive roots to arbitrary ones, see [V04, §4^o]. An ordered pair $\{r_1, s_1\}$ of roots is called *extraspecial* if $\{r_1, s_1\}$ is special and if for all special pairs $\{r, s\}$ with $r + s = r_1 + s_1$ we have $r_1 \preceq r$. Extraspecial pairs are in one-to-one correspondence with the roots in $\Phi^+ - \Delta$, where Φ^+ is the set of the positive roots, Δ is the set of the simple roots, see [Ca72, p. 58-59]. If $\{r, s\}$ is a pair of roots which is special but not extraspecial, then there is unique extraspecial pair $\{r_1, s_1\}$ such that

$$r + s = r_1 + s_1. \tag{3.1}$$

Four roots $\{r, s, -r_1, -s_1\}$ satisfies to (1.4), and

$$0 \prec r_1 \prec r \prec s \prec s_1. \tag{3.2}$$

Let $\{r_1, s_1\}$ be an extraspecial pair, and $\{r, s\}$ be a special pair satisfying to (3.1) and (3.2). We call these four roots $\{r_1, r, s, s_1\}$ the *Carter quartet*.

Proposition 3.1. [Ca72, Prop. 4.2.2]. *The signs of the structure constants $N(r, s)$ may be chosen arbitrarily for extraspecial pairs $\{r, s\}$, and then the structure constants for all pairs are uniquely determined.*

If the extraspecial pairs are known, then the structure constants are known and the Lie algebra is uniquely determined. In terms of Carter quartets, Carter's formula (1.4) can be rewritten as follows:

Lemma 3.2. *Let $\{r_1, r, s, s_1\}$ be a Carter quartet in any multiply-laced root system. By* (3.1) *roots $\{r_1, r, s, s_1\}$ satisfy to* (1.4)*, which can be transformed as follows:*

$$N(r,s) = \frac{|r+s|^2}{N(r_1, s_1)} \left(\frac{N(s - r_1, r_1) N(s_1 - r, r) |s_1 - r|^2}{|s|^2 |s_1|^2} + \frac{N(r_1, r - r_1) N(s_1 - s, s) |r - r_1|^2}{|r|^2 |s_1|^2} \right) \tag{3.3}$$

4. Carter quartets for root systems B_n

Theorem 4.1. *In the root system B_n, for any Carter quartet $\{r_1, r, s, s_1\}$, vectors $s - r_1$ and $r - r_1$ cannot both be roots.*

Theorem 4.2. *In the root system B_n, any Carter quartet $\{r_1, r, s, s_1\}$ has the following properties:*

(i) *If $s - r_1$ (resp. $r - r_1$) is a root, then $|s - r_1| = |s|$ (resp. $|r - r_1| = |r|$) .*

(ii) *The roots $r_1 + s_1$ and s_1 have the same length.*

Corollary 4.3. *In the root system B_n, the formula* (3.3) *looks as follows:*

$$N(r,s) = \frac{N(s-r_1,r_1)N(s_1-r,r) + N(r_1,r-r_1)N(s_1-s,s)}{N(r_1,s_1)}. \quad (4.1)$$

Relation (4.1), in fact, coincides with formula (2.1) used to calculate the structure constants in the simply-laced case.

5. Carter quartets for root systems C_n

The properties of Carter quartets given in Theorems 4.1 and 4.2 for B_n are not true for C_n, but are very close.

Theorem 5.1. (i) *In the root system C_n, for any Carter quartet $\{r_1, r, s, s_1\}$, vectors $s - r_1$ and $r - r_1$ are both roots if and only if $(r_1, s_1) = 0$. This condition does not depend on the pair $\{r, s\}$.*

(ii) *The number of extraspecial pairs $\{r_1, s_1\}$ with $(r_1, s_1) = 0$ is $n - 1$.*

(iii) *The inner product $(r_1, s_1) = 0$ if and only if*

$$|r_1 + s_1| = \sqrt{2}, \quad |r_1| = |s_1| = 1, \quad \textit{and} \quad \frac{|r_1+s_1|^2}{|s_1|^2} = 2.$$

Theorem 5.2. *In the root system C_n, for any Carter quartet $\{r_1, r, s, s_1\}$, if $s - r_1$ (resp. $r - r_1$) is a root then $|s - r_1| = |s|$ (resp. $|r - r_1| = |r|$).*

Corollary 5.3. *In the root system C_n, the formula* (3.3) *looks as follows:*

$$N(r,s) = \frac{|r_1+s_1|^2}{|s_1|^2}\left(\frac{N(s-r_1,r_1)N(s_1-r,r) + N(r_1,r-r_1)N(s_1-s,s)}{N(r_1,s_1)}\right). \quad (5.1)$$

The parameter $\varphi = |r_1 + s_1|^2/|s_1|^2$ in (5.1) takes values $1/2, 1, 2$. Note that φ depends only on the extraspecial pair $\{r_1, s_1\}$.

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Sandwich classification theorem by generic element method

Alexei Stepanov

The talk is about the lattice of subgroups of a Chevalley group $G(R)$ over a commutative ring R, containing the elementary subgroups $D(R)$ of another Chevalley group over the same ring. The standard description of the lattice asserts that it splits into a disjoint union of «sandwiches», parametrized by ideals of R. For example, the standard description in the case of $G = SL_n$ and D being the elementary subgroup of a classical group in the natural representation, was obtained by Nikolai Vavilov and Victor Petrov in 2000-2007. The most difficult step in the proof is extraction of a nontrivial elementary unipotent element. Using "decomposition of transvections" guarantees that the extracted unipotent is nontrivial. When "decomposition of transvections" does not work, this is a very difficult problem. Our method is to extract a unipotent from the generic element of the group scheme. It is easy to check that it is nontrivial. If it vanishes for all homomorphisms, sending the generic element to the elements of a considered subgroup H, then H lies in a closed subscheme. Using the result over fields and for subradical subgroups we show that this is impossible. Thus, H contains an elementary root unipotent as required.

The talk is based on joint work with Roman Lubkov.

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Работы Н.А.Вавилова по истории и философии математики

Сергей Пилюгин

Вавилов – историк математики

В 2020 – 2022 гг. Н.А. Вавилов опубликовал серию статей [1–6], объединенных общим заголовком "Компьютер как новая реальность математики". Сам автор несколько ограничивает значение этих текстов (или, по крайней мере, их части), говоря, что они имеют не научный и не исторический, а именно методический и методологический характер [2, с. 8]. На самом деле, статьи [2–6] – очень серьезное и глубокое историческое исследование некоторых классических задач теории чисел, написанное первоклассным математиком. В них обсуждаются проблема Варинга в различных постановках, проблема Гольдбаха, задачи, связанные с поведением чисел Мерсенна и Ферма и их аналогов. Приведено огромное количество указаний на результаты, полученные в этой области как профессиональными математиками, так и любителями (среди которых школьные учителя, сельские священники и даже генералы).

Большое внимание уделено точным постановкам классических задач; как показывает Н.А., в большинстве исторических обзоров и книг эти постановки отражены неточно. Точные постановки подтверждаются буквально чтением оригиналов "под увеличительным стеклом". Вот что Н.А. пишет в [4] на стр. 10 о маргиналии Гольдбаха в письме к Эйлеру, в которой сформулирована его гипотеза: "В более крупном разрешении хорошо видно, что слова "die grosser ist als 1" дописаны вообще без пробелов под строкой, потом 1 там заменена на 2, а потом снова на 1".

Конечно, в соответствии с названием цикла, Н.А. детально прослеживает фантастический прогресс в теории чисел, связанный с использованием современных компьютеров. Например, в статье [4] обстоятельно проанализировано полное решение нечетной проблемы Гольдбаха (каждое нечетное натуральное число $n > 5$ можно представить как сумму трех натуральных простых)

Х. Хельфготтом, опубликованное в 2013 – 2014 гг. и соединившее достижения, основанные на классических подходах, с существенным использованием компьютеров.

Значительная часть текстов посвящена методам теории чисел; так, автор приводит большое количество полиномиальных тождеств, использованных различными математиками начиная с Эйлера и Лиувилля. Он в деталях обсуждает идеи и подходы, позволившие превратить современную теорию чисел в важнейший отдел математики.

Н.А. включил в тексты много задач, которые читатель может решать, используя компьютер.

Статьи написаны человеком замечательной образованности (и, конечно, рассчитаны на очень образованного и вдумчивого читателя); тексты включают цитаты (данные без перевода на русский язык) на латыни, английском, немецком, французском языках, на той своеобразной смеси немецкого и латыни, на которой переписывались Эйлер и Гольдбах, в них встречаются отрывки на итальянском, польском, греческом ...

Для Н.А. прогресс математики был неотъемлемой частью развития всей культуры человечества, поэтому совершенно естественными выглядят упоминания в его текстах Блаженного Августина и Леонардо да Винчи, философа О. Шпенглера, писателей братьев Гримм, О. Уайльда, Л. Кэрролла и Л. Борхеса, композиторов И.-С. Баха, Генделя, Скарлатти, Джустини, Рамо и путешественника Марко Поло (и даже художника Васи Ложкина).

К 300-летию С.-Петербургского университета журнал Вестник СПбГУ начал публиковать серию исторических обзоров о достижениях петербургских и ленинградских математиков, связанных с университетом. Для этой серии Н.А. Вавилов начал писать цикл статей, посвященных вкладу петербургских математиков в теорию линейных, классических и алгебраических групп. Вышли две статьи [7,8] из задуманного цикла из четырех статей; к сожалению, уход Н.А. из жизни не позволил ему завершить этот замысел.

Вавилов – философ математики

Статья [1] содержит большой раздел, в котором Н.А. глубоко философски анализирует те фундаментальные изменения в математике, которые явились следствием применения современных компьютеров. Содержание этого раздела наглядно характеризуют заголовки его абзацев: контакт с реальностью, соотношение идей и вычислений, конечное и бесконечное, возможное и невозможное, проблема промежуточных размеров, детерминированное и случайное, выводимое и наблюдаемое, алгоритмическое мышление, связь времен, теоретическая и экспериментальная математика.

Очень важной частью философского наследия Н.А. является его программная статья [9], содержащая нетривиальный анализ соотношения логики и интуиции в процессе развития математики.

Вначале он формулирует несколько (считающихся общепринятыми) тезисов о структуре математического доказательства; мы приводим, значительно упрощая, лишь некоторые из них:

• доказательство – это формальный текст, в котором по строго определенным правилам результат выводится из набора аксиом и ранее полученных результатов;

• иногда очень трудно предъявить доказательство, но его проверка – это чисто технический процесс (в частности, доступный компьютеру);

• существуют общепринятые во всех областях математики критерии строгости доказательства;

• все утверждения, формулируемые в учебных курсах достаточно продвинутого уровня, приводятся с полными и четкими доказательствами.

Приводя очень убедительные примеры и аргументы, Н.А. опровергает основную часть сформулированных выше тезисов. Согласно его концепции, доказательство более или менее содержательного математического утверждения – это не формальный вывод результата из аксиом и предшествующих теорем, а скорее "дорожная карта", пользуясь которой (и прикладывая при этом иногда не менее усилий, чем приложил автор доказательства для его создания), математик-профессионал может убедиться в верности доказываемого результата.

Именно на этом этапе проверки нового знания и требуется разумное сочетание логики и интуиции, взаимодействующих на базе фундаментальной математической подготовки.

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Николай Вавилов и МАТЕМАТИКА для нематематиков

Николай Васильев, Людмила Вьюненко, Владимир Халин and Александр Юрков

Профессор Николай Александрович Вавилов (1952-2023) был энциклопедически образованным человеком, много знающим в разных областях науки, культуры, искусства, свободно говорил на нескольких европейских языках и мог читать на многих языках мира, включая китайский, санскрит и фарси. Поражала его работоспособность: под настроение Николай Александрович мог за несколько дней написать полноценную статью страниц на тридцать, да еще на великолепном английском – именно так родилась «The Skies are Falling: Mathematics for Non-Mathematicians» [1]. К математике Николай Александрович относился как к «...высшему проявлению человеческого духа и культуры, ценным независимо от каких-либо приложений». Именно этим он объяснял совершенно особую роль математического образования в функционировании общества, выделяя три принципиально разных уровня: до-университетский; математика для математиков; математика для нематематиков. Николай Александрович считал, что «...самый важный аспект преподавания математики на элементарном уровне – выработка интеллектуальной честности, т.е. способности отличать то, что ты понимаешь, от того, чего ты не понимаешь; то, что имеет точный смысл, от того, что не имеет; то, что сказано, от того, что имеется в виду; возможное от невозможного; истинное от ложного; доказанное от предполагаемого». Второй столь же важный аспект – физкультура мозга, подготовка к умению решать любые трудные задачи. На университетском уровне на первый план выходят другие цели – в первую очередь развитие математического способа мышления, т.е. способности начинать с первых принципов, рассматривать самый простой случай, использовать аналогии и метафоры, обобщать и специализировать, и так далее. Ну и, конечно, развитие собственно математического понимания и тренировка основных способов рассуждений. Профессор Н.А. Вавилов является автором уникальной концепции обучения математике нематематиков на университетском уровне с использованием систем символьных вычислений и компьютерной алгебры. Николай Александрович считал, что «преподавание математики должно интриговать, увлекать и очаровывать» и предлагал учить математике по-новому – перепоручить основную часть рутинных вычислений системам компьютерной алгебры и целиком сфокусироваться на идейной стороне математики, делая акцент на основные, самые важные, полезные, интригующие и увлекательные пласты математики – понятия, идеи, аналогии, конструкции и метафоры. Он настаивал, что нематематиков нужно «учить математике так, как мы, математики, ее понимаем т.е. в первую очередь, ПОНИМАНИЮ» [1]. Начиная с 2005 года профессор Н.А. Вавилов более 10 лет читал на экономическом факультете СПбГУ авторский двухсеместровый курс «Математика и компьютер» для студентов специальностей «Математические методы в экономике» и «Прикладная информатика в экономике». Концепция профессора Вавилова Н.А. состояла в том, чтобы сфокусироваться исключительно на понимании и больших идеях, заменяя значительную часть доказательств и фактических навыков – кроме самых основных и тех, которые необходимы для понимания, – на компьютерные вычисления, эксперименты и визуализацию. Трудная часть работы по реализации этого курса состояла в том, чтобы создать специальную систему из нескольких сотен учебных задач, которые требуют одновременно математического и алгоритмического мышления [1]. Наработки по курсу «Математика и компьютер», лишь часть которых удалось «уложить» в сжатый формат учебника «Mathematica для нематематика» [3], по оценкам самого профессора Вавилова Н.А. насчитывали более тысячи страниц! Курс концентрировался на основных математических идеях, а не на специфических приложениях. Вот как сам Николай Александрович описывал эту работу: «Вначале мы излагали какие-то новые математические понятия и идеи, а также формулировали несколько ключевых утверждений, иногда с набросками доказательств. Полностью доказательства излагались только в тех случаях, когда они были особенно короткими и наглядными или содержали мощные общие соображения, полезные во многих ситуациях. После этого мы давали рекомендации для дальнейшего чтения, для тех, кто хотел глубже изучить эти понятия, и переходили к алгоритмам и компьютерным демонстрациям, вычислениям, графике и т.д. ... При активном участии и интересе со стороны студентов нам удалось покрыть за то же время гораздо больше математики, более разнообразной математики, более интересной и, в конечном счете, более полезной математики с гораздо лучшими результатами, чем это было бы возможно при более традиционном подходе» [2]. По результатам многолетней практики преподавания курса «Математика и компьютер» был опубликован учебник «Mathematica для нематематика» [3]. Учебник был поддержан грантом Фонда Потанина, а в 2021 году победил в номинации «Лучшая инновационная идея» в конкурсе инновационных проектов в сфере науки и высшего образования Правительства Санкт-Петербурга. Студентам подход профессора Вавилова весьма импонировал, и даже по прошествии лет они с воодушевлением вспоминают занятия по его курсу: «Математика и компьютеры» – одна из дисциплин обучения на специальности «Прикладная информатика в экономике» в СПбГУ, и соответствующая ей книга оставила приятное послевкусие, поскольку концепции и алгоритмы, описанные в книге максимально актуальны при решении любых задач, где требуется математика, а в работе аналитика или профессионала в области данных математика необходима повсеместно» (Валиотти Николай, выпуск 2008); «Mathematica для нематематика» – это единственная университетская книга, к которой я периодически возвращаюсь даже спустя десять лет после выпуска» (Аплеев Даниил, выпуск 2011). Удивляла и восхищает способность Николая Александровича с легкостью использовать информационные технологии в преподавании классической и современной математики как студентам, так и профессионалам. Профессор Н.А. Вавилов является автором более двадцати авторских общих и специальных онлайн-курсов для студентов и аспирантов, изучающих современную математику и компьютерные науки. Среди них «Алгебра», «Высшая алгебра», «Компьютерная алгебра», «Алгебраическая геометрия», «Алгебраические группы», «Алгебры и группы Ли», «Центральные простые алгебры», «Алгебры Хопфа и теория Галуа», «Высшие законы композиции», «Модулярные представления конечных групп», «Алгебры Клиффорда и спинорные группы», «Исключительные объекты в алгебре и геометрии», «Английский язык для математиков», «Domino Tilings», «Jacobian Conjecture» (сайт факультета МКН СПбГУ: https://math-cs.spbu.ru/courses/required/ и https://math-cs.spbu.ru/courses/special/ + сайт лекториум: https://www.lektorium.tv/speaker/3130).

Заменить Николая Александровича невозможно. Его талант, невероятное трудолюбие, увлечённость, ответственность, его особый стиль преподавателя и учёного, невероятная харизма снискали ему глубокое уважение учеников и коллег. Память о профессоре Николае Александровиче Вавилове будет жить в сердцах всех, кому посчастливилось встретиться с этим удивительным человеком на жизненном пути.

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