Gröbner-Shirshov bases, Dehn's algorithm and small cancellation rings

Agatha Atkarskaya

Hebrew University of Jerusalem
Einstein Institute of Mathematics

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- To answer the above question it is natural to define the object using generators and defining relators;
- However, there can be unexpected corollaries of the initial relators and the final object may become trivial;
  Adian-Rabin theorem states that there is no algorithm that takes as an input a finite group presentation and decides if the given group is trivial.
- Which kinds of presentations are good?

Small cancellation groups

Let $G = \langle x_1, \ldots, x_n \mid R_1, \ldots, R_k, \ldots \rangle$ be a group, where all R_i are cyclically reduced, the list of relators is closed under taking inverses and cyclic shifts of relations.

If $R_i \neq R_j$ and $R_i = cR_i'$, $R_j = cR_j'$, then c is called a *small piece*.

We say that G satisfies condition $\mathrm{C}(m)$ if every R_i can not be written as a product of less than m small pieces.

If $m \geqslant 7$, then G is hyperbolic.

Small cancellation groups

Theorem (Geendlinger's Lemma)

If G satisfy $\mathrm{C}(7)$ and $A = 1$ in G, then $A = LuR$, where $uu' = R_i$ for some i and $|u| > \frac{1}{2}|R_i|$.

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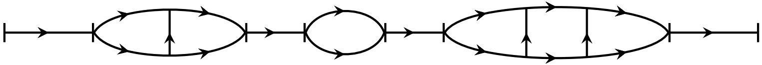
It $A = B$ in such G and A, B do not contain $> \frac{1}{2}$ of any R_i, then A, B can be viewed as paths in a one-layer map.



Rips construction

There exist subgroups of small cancellation groups with exotic properties.

Theorem (E. Rips, 1982)

There is a finitely presented small cancellation group G such that:

- *G has finitely generated subgroups whose intersection is not finitely generated.*
- *G has a finitely generated but not finitely presented subgroup.*
- *The subgroup membership problem in G is not solvable.*

Iterated small cancellation theory

Given $G = \langle x_1, \dots, x_n \mid R_1, \dots, R_t, \dots \rangle$, our goal not small cancellation conditions, our real goal is Greendlinger's Lemma.

There are many group presentations with the following structure:

$$\{R_1, \dots, R_t, \dots\} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \dots,$$

where every $\mathcal{R}_i$ satisfies $C(m)$ and moreover $\mathcal{R}_i$ satisfies $C(m)$ modulo $\mathcal{R}_j$, $j \leqslant i$. Then every intermediate quotient $\mathrm{F}/\langle\langle R_1, \dots, R_i \rangle\rangle$ satisfies Greendlinger's Lemma. Thus, G satisfies it as well.

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- The Burnside group (f.g. group with identity $x^n = 1$) is infinite for odd $n \geqslant 557$ (S. Adian and P. Novikov, A. Olshanskii, E. Rips, K. Tent, A.A.), and even $n \geqslant 8000$ (S. Ivanov, I. Lysenok).
- Construction of Tarskii monster group (infinite group with all proper subgroups cyclic of the same order) by A. Olshanskii.

Constructing rings with exotic properties

This is a joint work with A. Kanel-Belov, E. Plotkin and E. Rips.

Our initial motivation is the following old problem in Ring Theory (which is still not solved):

Problem (1970s, Kaplansky, L'vov, Latyshev)

Does there exist a division algebra infinite dimensional over its center with finitely generated multiplicative group?

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Our general idea is as follows. We want to take such a quotient of $\mathbb{Z}_2\mathrm{F}$, F is a non-abelian free group, that it is infinite dimensional over its center and its every element is equal to a monomial.

Clearly, it is sufficient that every binomial $1 + w$ is equal to a monomial. So, we enumerate all elements of F, $\{w_1, w_2, \ldots\}$, and impose the relations $1 + w_i = v_i$ on $\mathbb{Z}_2\mathrm{F}$. Now the question is if $\mathbb{Z}_2\mathrm{F}/\mathcal{I}$, where $\mathcal{I} = \langle\langle 1 + w_i = v_i; i = 1, 2, \ldots\rangle\rangle$, is infinite dimensional over its center (or at least non-trivial).

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Notice that it is easy to produce a quotient of $\mathbb{Z}_2\langle x, y\rangle$ that is infinite dimensional over $\mathbb{Z}_2$ and with every element equal to a monomial (everything what we want except inverting).

Gröbner-Shirshov bases of polynomial ideals

We fix deglex order on monomials. Let $I = \langle\langle f_1, f_2, \ldots\rangle\rangle \triangleleft k\langle x_1, \ldots, x_m\rangle$. Then $f_1, f_2, \ldots$ is a Gröbner-Shirshov basis of I if for every $Q \in I$ the leading monomial of Q is of the form $L\overline{f}_i R$, where $\overline{f}_i$ is the leading monomial of f_i.

So $Q \in I$ can be transformed to 0 by the greedy algorithm, which consists of steps: $Q \longmapsto Q - Lf_iR$, etc.. (the steps are called *reductions*).

If $L_1 f_i R_1$ and $L_2 f_j R_2$ have the same highest monomials, then $S(f_i, f_j) = L_1 f_i R_1 - L_2 f_j R_2$ is called *s-polynomial*.

Theorem

$\{f_1, f_2, \ldots\}$ is a Gröbner-Shirshov basis of the ideal I if and only if all s-polynomials of f_i, f_j can be reduced to 0.

We consider words of the form $u_i = x^2 y^i xy$ for $i > 0$. Now we enumerate all pairs of words in the semigroup $\langle x, y \rangle$ and impose the relations $w_j + w_k = u_{i(j,k)}$ on $\mathbb{Z}_2\langle x, y \rangle$, where we choose $i(j,k) > 2(|w_j| + |w_k|)$. Then $u_{i(j,k)}$ is the leading monomial of $w_j + w_k = u_{i(j,k)}$. Clearly, u_s, u_t do not have overlaps for $s \neq t$. Therefore, $\{w_j + w_k = u_{i(j,k)}\}_{j,k}$ is a Gröbner-Shirshov basis of the corresponding ideal I.

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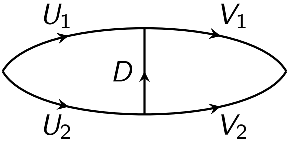
Now let us add inverses. Namely, take $\mathbb{Z}_2\langle x, y, a, b\rangle$, take the same $w_j + w_k = u_i$ and add $xa = ax = yb = by = 1$. Then we have overlaps in the set of the highest monomials. Take the s-polynomial for $ax - 1$ and $u_i - w_j - w_k$:

$$a(u_i - w_j - w_k) - (ax - 1)xy^i xy = xy^i xy - aw_j - aw_k.$$

The leading monomial $xy^i xy$ has an overlap with another u_t. Adding more and more s-polynomials produces more and more overlaps of the highest monomials.

Deglex ordering does not work well for "group-like" relations.

Let $G = \langle x, y \mid R_1, \ldots, R_n \rangle$ be given by a finite symmetrised Dehn's presentation. Then $kG \cong k\mathrm{F}/I$, where $I = \langle\langle R_1 - 1, \ldots, R_n - 1 \rangle\rangle$. Consider the following set of generators of I: $\{A_{ij} - B_{ij}\}_{i,j}$, where $A_{ij} B_{ij}^{-1} = R_i$. It is not a Gröbner-Shirshov basis of I with respect to deglex order.



$$R_s = U_1 D^{-1} V_1^{-1}$$

$$R_t = V_1 V_2^{-1} D$$

$U_1 U_2 - V_1 V_2 = 0$ in kG, but it do not contain leading monomials of any relators $A_{ij} - B_{ij}$. Notice that this can happen even if G satisfy $C(7)$.

Small cancellation rings

Let $\mathcal{I} = \langle\langle p_i = \sum_{j=1}^{n_j} \alpha_{ij} m_{ij}; i \geqslant 1 \rangle\rangle \lhd k\mathrm{F}$, $\mathcal{R} = \{p_i; i \geqslant 1\}$ and $\mathcal{M} = \{m_{ij}\}_{i,j}$.

Condition (Compatibility Condition)

- *For every* $\alpha \in k$ *we have* $\alpha \cdot p_i \in \mathcal{R}$.
- *If* $m_{ij} = xm'_{ij}$, *then* $x^{-1} \cdot p_i \in \mathcal{R}$.
- *If* $m_{ij} = m'_{ij}x$, *then* $p_i \cdot x^{-1} \in \mathcal{R}$.

Small pieces for rings

Let $C \in \mathcal{M}$,

$$p = \sum_{i=1}^{m_1} \alpha_i a_i + A_1 C A_2 \in \mathcal{R},\ \ q = \sum_{i=1}^{m_2} \beta_i b_i + B_1 C B_2 \in \mathcal{R}.$$

If at least one of $B_1 \cdot A_1^{-1} \cdot p$ and $p \cdot A_2^{-1} \cdot B_2$ does not belong to $\mathcal{R}$ (even after the cancellations), then C is called *a small piece with respect to* $\mathcal{R}$

We require that 1 is a small piece.

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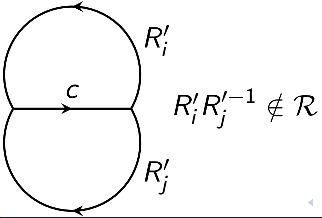
There exists a generalization of small pieces (in the group sense) called *graph small cancellation condition*. The above definition follows similar intuition.



Small cancellation condition for rings

Condition (Small Cancellation Condition)

Assume $q_1, \dots, q_n \in \mathcal{R}$ *and a linear combination* $\sum_{j=1}^{n} \gamma_j q_j$ *is non-zero after additive cancellations. Then there exists a monomial A in* $\sum_{j=1}^{n} \gamma_j q_j$ *with a non-zero coefficient such that*

- *either A can not be represented as a product of small pieces,*
- *or every representation of A of a form* $A = c_1 \cdots c_l$*,*
  where $c_1, \dots, c_l$ *are small pieces,*
  contains at least 11 small pieces.

Structure of small cancellation rings

"Informal structure theorem"

Theorem (A.A., A. Kanel-Belov, E. Plotkin, E. Rips, 2021)

Let $k\mathrm{F}/\mathcal{I}$ be a small cancellation ring, $\mathcal{I} = \langle\langle p_i; i \geqslant 1\rangle\rangle$. Then it is non-trivial, its structure can be described in terms of "multi" one-layer maps, and $\{p_i\}_{i\geqslant 0}$ is a Gröbner-Shirshov basis of $\mathcal{I}$ with respect to a special monomial order.

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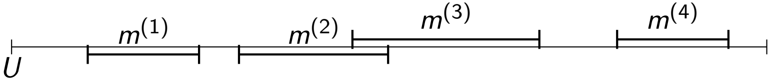

$m^{(i)}$ are maximal occurrences of elements from $\mathcal{M}$ in a word U. They are used to define a complexity of U and the order on monomials is based on the complexity.

Small cancellation rings: examples

If $G = \langle x_1, \ldots, x_n \mid R_1, \ldots, R_t, \ldots \rangle$, then $kG \cong k\mathrm{F}/\mathcal{I}$, where $\mathcal{I} = \langle\langle R_i - 1; i \geqslant 1 \rangle\rangle$. If G satisfies $\mathrm{C}(m)$, $m \geqslant 22$, then kG is a small cancellation ring.

Notice that the above set of generators of $\mathcal{I}$ itself does not satisfy small cancellation conditions. We need to extend it.

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Let $\mathcal{I} = \langle\langle 1 + w = v \rangle\rangle \lhd k\mathrm{F}$, where $v = x^\alpha y x^{\alpha+1} y \ldots y x^\beta y$, and $|w| \ll \alpha \ll \beta$. Then $k\mathrm{F}/\mathcal{I}$ is a small cancellation ring.

Again we need to extend the generating set to fulfil small cancellation conditions. Small pieces here are subwords of $w^{\pm N}$ and subwords of $v^{\pm 1}$ that contain less than two letters $y^{\pm 1}$.

Further research

- Develop iterated small cancellation theory for rings.
- Study further properties of small cancellation rings.
  - Small cancellation rings are non-amenable and contain free associative subalgebras (A.A., 2023).
  - Prove generalisation of Bergman's centraliser theorem (N. Miasnikov gave a partial description of centralisers in $k\mathrm{F}$).
- Develop geometric theory for rings in parallel to geometric group theory.