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Evaluations of non-commutative polynomials on finite dimensional algebras.

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We survey what is known about evaluations of a polynomial p in several non-commuting variables taken in a matrix algebra $M_n(K)$ over a field.
It has been conjectured that for any n, when p is multilinear, the image of p is either zero, or the set of scalar matrices, or the set $sl_n(K)$ of matrices of trace 0, or all of $M_n(K)$.
The conjecture is true for $n = 2$ and K being

- any quadratically closed field. (Kanel-Belov, Malev, Rowen)
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Nonmultilinear polynomials

Although the analogous assertion fails for completely homogeneous polynomials, one can salvage the conjecture in this case by including the set of all non-nilpotent matrices of trace zero and also permitting dense subsets of $M_2(K)$.

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Kaplansky's question

We consider the version for algebras, reputedly raised by Kaplansky, of the possible image set $\operatorname{Im} p$ of a polynomial p on matrices. When $p = x_1x_2 - x_2x_1$, its image consists of all matrices of trace 0, by a theorem of Albert and Muckenhoupt.

Over a finite field K, Kaplansky's question was settled for arbitrary polynomials by Chuang, who proved that a subset $S \subseteq M_n(K)$ containing 0 is the image of a polynomial with constant term zero, if and only if S is invariant under conjugation.

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Definition

A polynomial $p \in K\langle x_1, \dots, x_m\rangle$ is called **multilinear** of degree m if it has degree 1 in each variable. Thus, a polynomial is multilinear if it is of the form

$$p(x_1, \dots, x_m) = \sum_{\sigma \in S_m} c_\sigma x_{\sigma(1)} \cdots x_{\sigma(m)},$$

where S_m is the symmetric group in m letters and $c_\sigma \in K$.

Linear span of the image.

At first glance, our question looks rather easy. Over an infinite field, one can ascertain the linear span of the values of multilinear polynomial. The linear span of its values comprise a Lie ideal since

$$[a, p(a_1, \dots, a_n)] = p([a, a_1], a_2 \dots, a_n) + p(a_1, [a, a_2] \dots, a_n) + \cdots .$$

Herstein characterized Lie ideals of a simple ring R as either being contained in the center or containing the commutator Lie ideal $[R, R]$.

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Lvov formulated Kaplansky's question as follows:
Let p be a multilinear polynomial over a field K. Is the set of values of p on the matrix algebra $M_n(K)$ a vector space?

An explicit version:
Conjecture: If p is a multilinear polynomial evaluated on the matrix ring $M_n(K)$, then $\operatorname{Im} p$ is either $\{0\}$, K, $sl_n(K)$, or $M_n(K)$. Here K is the set of scalar matrices and $sl_n(K)$ is the set of matrices of trace zero.

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Im (p) can really be

- $\{0\}$, let p be any PI, in particular s_{2n} (see Amitsur-Levitzky theorem)
- the set $\mathrm{sl}_n(K)$ of matrices of trace 0, let $p(x,y) = [x,y]$
- $M_n(K)$, let $p(x) = x$
- or the set of scalar matrices

Examples

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Problem

What is the smallest possible degree $c(n)$ of a multilinear central polynomial identity in the algebra of $n \times n$ matrices?

Formanek (1972): an example of degree n^2.
Razmyslov (1973): an example of degree $3n^2 - 2$.
Halpin(1983): Uses Razmyslov method to construct an example of degree n^2.
Thus $c(n) \leq n^2$.
$c(1) = 1$ and $c(2) = 4$
Drensky, Kasparian (1983): $c(3) \geq 8$ (1985): $c(3) = 8$

Conjecture (Formanek (1991))

$c(n) = \frac{1}{2}(n^2 + 3n - 2)$

Drensky, Piacentini Cattaneo (1994): $c(4) \leq 13$.
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The case $n = 2$

Theorem (Malev)

If p is a multilinear polynomial evaluated on 2×2 matrices with entries in the arbitrary field K, then $\operatorname{Im} p$ satisfies one of the following:

- $\operatorname{Im} p = \{0\}$,
- $\operatorname{Im} p$ *is the set of scalar matrices, or*
- $\operatorname{Im} p \supseteq sl_2(K)$

Theorem (Malev)

Let $p(x_1, \ldots, x_m)$ be a multilinear polynomial evaluated on 2×2 matrices with real entries. Then $\operatorname{Im} p$ is one of the following:

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Theorem (Kanel-Belov, Malev, Rowen)

If p is a homogeneous Lie polynomial evaluated on the matrix ring $M_2(K)$ (where K is an algebraically closed field), then $\operatorname{Im} p$ *is either* $\{0\}$, *or* K *(the set of scalar matrices), or the set of all non-nilpotent matrices having trace zero, or* $\mathrm{sl}_2(K)$, *or* $M_2(K)$.

Example (Kanel-Belov)

Take $h(u_1, \ldots, u_8)$ central on 3×3 matrices. For the 2×2 matrices $x_1, \ldots, x_9$ we consider the homogeneous Lie polynomial $p(x_1, \ldots, x_9) = h(\mathrm{ad}_{[[[x_1,x_9],x_9],x_9]}, \mathrm{ad}_{x_2}, \mathrm{ad}_{x_3}, \ldots, \mathrm{ad}_{x_8})(x_9)$.

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The case $n = 3$

Theorem (Kanel-Belov, Malev, Rowen)

If p is a multilinear polynomial evaluated on 3×3 matrices over algebraically closed field then $\operatorname{Im} p$ is one of the following:

- $\{0\}$,
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- *a dense subset of* $M_3(K)$,
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The case $n \geq 4$

Theorem (Kanel-Belov,Malev,Rowen)

If p is a multilinear polynomial evaluated on 4×4 matrices, over algebraically closed field, neither PI nor central then $\dim \operatorname{Im} p \geq 14$, *equality holding only if any value of p has eigenvalues* $(\lambda_1, \lambda_2, -\lambda_1, -\lambda_2)$.

Theorem (Kanel-Belov,Malev,Rowen)

If p is a multilinear polynomial evaluated on $n \times n$ matrices, over algebraically closed field, neither PI nor central then $\dim \operatorname{Im} p \geq n^2 - n + 3$.

The case $n \geq 4$

Theorem (Kanel-Belov,Malev,Rowen)

If p is a multilinear polynomial evaluated on 4×4 *matrices, over algebraically closed field, neither PI nor central then* $\dim \operatorname{Im} p \geq 14$, *equality holding only if any value of p has eigenvalues* $(\lambda_1, \lambda_2, -\lambda_1, -\lambda_2)$.

Theorem (Kanel-Belov,Malev,Rowen)

If p is a multilinear polynomial evaluated on $n \times n$ *matrices, over algebraically closed field, neither PI nor central then* $\dim \operatorname{Im} p \geq n^2 - n + 3$.

Quaternion algebra

Theorem (Malev)

Let $p(x_1, \ldots, x_m)$ be a multilinear polynomial evaluated on quaternion algebra Then $\operatorname{Im} p$ *is one of the following:*

- *$\{0\}$,*
- *the set of scalars,*
- *the set of vectors, or*
- *the set of all quaternions.*

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Theorem (Kanel-Belov, Malev, Pines, Rowen)

If p is a multilinear polynomial evaluated on the Octonion algebra $\mathbb{O}$, then $\operatorname{Im} p$ is either $\{0\}$, or $\mathbb{R} \subseteq \mathbb{O}$ (the space of scalar octonions), or $V \subseteq \mathbb{O}$ – the space of pure octonions, or $\mathbb{O}$.

We also classify possible images of semi-homogeneous polynomials:

Theorem (Kanel-Belov, Malev, Pines, Rowen)

If p is a semihomogeneous non-commutative non-associative polynomial of weighted degree $d \neq 0$ evaluated on the the octonion algebra $\mathbb{O}$, then $\operatorname{Im} p$ is either $\{0\}$, or $\mathbb{R}$, or $\mathbb{R}_{\geq 0}$, or $\mathbb{R}_{\leq 0}$, or V, or some Zariski dense subset of $\mathbb{O}$.

Octonion algebra

Theorem (Kanel-Belov, Malev, Pines, Rowen)

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Definition

Malcev algebra is is a nonassociative anticommutative algebra that satisfies the Malcev identity:

$$(xy)(xz) = ((xy)z)x + ((yz)x)x + ((zx)x)y.$$

We will consider the Malcev algebra $(V, [\cdot, \cdot])$.

Theorem (Kanel-Belov, Malev, Pines, Rowen)

Let $p(x_1, \ldots, x_m)$ be **arbitrary** *polynomial in m anticommutative variables. Then its evaluation on V can be either $\{0\}$ or V.*

Proof.

$\|p(x_1, \ldots, x_m)\|^2$ is a polynomial in $7m$ variables.

It is either

- 0
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Main theorem

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□

Example

p can be PI of V:

$$p(x_1, x_2, \dots, x_{15}) = s_{14}(ad_{x_1}, ad_{x_2}, \dots, ad_{x_{14}})(x_{15}).$$

Remind: $ad_x(y) = [x, y]$,

$$s_n(x_1, \dots, x_n) = \sum_{\sigma \in S_n} (-1)^{\sigma} x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

Definition

A Jordan algebra is a non-associative algebra whose multiplication $\circ$ satisfies the following axioms:

1. $x \circ y = y \circ x$ (commutativity)
2. $(x \circ y) \circ (x \circ x) = x \circ (y \circ (x \circ x))$ (Jordan identity).

Definition

We define J_n (n-dimensional Jordan algebra) as follows:

- the base of J_n is the set $\{e_0 = 1, e_1, \ldots, e_{n-1}\}$,
- the product $\circ$ is defined as $1 \circ x = x \circ 1 = x$ for any x, $e_i \circ e_j = \delta_{ij}$.

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Examples

1. J_3: self-adjoint 2×2 real matrices with standard Jordan product $x \circ y = \frac{xy+yx}{2}$. Basis: 1 being the identity matrix, $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
2. J_4: self-adjoint 2×2 complex matrices. Basis: $\{e_0 = 1, e_1, e_2, e_3\}$, $e_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$.
3. J_6: self-adjoint 2×2 quaternionic matrices. Basis $\{e_0 = 1, e_1, \dots, e_5\}$, $e_4 = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}$, $e_5 = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}$.
4. J_{10}: self-adjoint 2×2 octonionic matrices. Basis $\{e_0 = 1, e_1, \dots, e_9\}$, $e_6 = \begin{pmatrix} 0 & l \\ -l & 0 \end{pmatrix}$, $e_7 = \begin{pmatrix} 0 & il \\ -il & 0 \end{pmatrix}$, $e_8 = \begin{pmatrix} 0 & jl \\ -jl & 0 \end{pmatrix}$, $e_9 = \begin{pmatrix} 0 & kl \\ -kl & 0 \end{pmatrix}$.

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Theorem (Malev, Yavich, Shayer)

Let p be any commutative non-associative polynomial. Then its evaluation on J_n is either $\{0\}$, or $\mathbb{R}$ (i.e. one-dimensional subspace spanned by the identity element), the space of pure elements V ($(n-1)$-dimensional vector space $\langle e_1, \ldots, e_{n-1} \rangle$) , or J_n.

Examples

1. The associator $p(x, y, z) = (xy)z - x(yz)$. $\operatorname{Im} p = V$.
2. We can construct the example of central polynomial as well: if $p(x, y, z)$ is an associator, then $p(x_1, y_1, z_1) \circ p(x_2, y_2, z_2)$ is central.
3. $p(p(x_1, y_1, z_1) \circ p(x_2, y_2, z_2), y_3, z_3)$ (where p is associator) is PI.

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Does there exist a multilinear 3-central polynomial evaluated on $M_3(K)$?

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Does there exist a Lie 3-central polynomial evaluated on $M_3(K)$?

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What can be the image set of a multilinear polynomial evaluated on $M_3(\mathbb{R})$?

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Classify all possible 15-dimensional image sets of a multilinear polynomial evaluated on $M_4(K)$ and 23- and 24-dimensional image sets of a multilinear polynomial evaluated on $M_5(K)$?

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Thank you!