On Grothendieck–Serre conjecture concerning principal bundles

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The conjecture

Let R be a regular local ring. Let $\mathbf{G}$ be a reductive group scheme over R. A well-known conjecture due to Grothendieck and Serre assertes that a principal $\mathbf{G}$-bundle over R is trivial, if it is trivial over the fraction field of R.

The conjecture was stated by J.-P.Serre in 1958 in so called constant case and by A.Grothendieck in 1968 in the general case.

The conjecture is solved in positive if R contains a field.

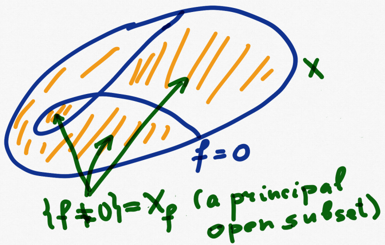
In the first part of the talk we will discuss smooth complex algebraic varieties and some examples to the conjecture in which as the group $\mathbf{G}$, so the principal $\mathbf{G}$-bundle are involved only tacitely (non-explicitly).

Some notation

In this introduction we give couple results motivating the conjecture in the constant case. To do that recall some notation.

Let X be an affine complex algebraic variety, smooth and irreducible. Let $\mathbb{C}[X]$ be the ring of regular functions on X and $f \in \mathbb{C}[X]$ be a non-zero function. Let

$$X_f := \{x \in X : f(x) \neq 0\}.$$



This open subset is called the principal open subset of X corresponding to the function f.

This open subset X_f is itself is an affine algebraic variety and its ring of regular functions $\mathbb{C}[X_f]$ is the localization $\mathbb{C}[X]_f$ of the ring $\mathbb{C}[X]$ with respect to the element f.

If A is a $\mathbb{C}[X]$-algebra, then we write A_f for the localization of A with respect to $f \in \mathbb{C}[X]$.

Serre's theorem (1958)

Let A be a $\mathbb{C}[X]$-algebra, which is a free finitely generated $\mathbb{C}[X]$-module of rank n. Suppose that A is isomorphic to the matrix algebra $M_r(\mathbb{C}[X])$ locally for the complex topology on X. Suppose further that for a non-zero function $f \in \mathbb{C}[X]$ the $\mathbb{C}[X_f]$-algebras

$$A_f \quad \text{and} \quad M_r(\mathbb{C}[X_f])$$

are isomorphic.

Then for any point $x \in X$ there is a regular function $g \in \mathbb{C}[X]$ such that $g(x) \neq 0$ and

$$A_g \cong M_r(\mathbb{C}[X_g])$$

as the $\mathbb{C}[X_g]$-algebras. **In the other words, the $\mathbb{C}[X]$-algebras**

$$A \quad \text{and} \quad M_r(\mathbb{C}[X])$$

are isomorphic locally for the Zariski topology on X.

Ojanguren's theorem (1982)

Let X and $\mathbb{C}[X]$ be as above and let $a_i, b_i \in \mathbb{C}[X]$ be invertible functions on X, where $i \in \{1, ..., r\}$. Consider two quadratic spaces

$$P := \Sigma_{i=1}^{r} a_i T_i^2 \text{ and } Q := \Sigma_{i=1}^{r} b_i T_i^2$$

over $\mathbb{C}[X]$. Suppose for a non-zero function $f \in \mathbb{C}[X]$ these quadratic spaces are isomorphic over the ring $\mathbb{C}[X_f]$.

Then the quadratic spaces

$$P \text{ and } Q$$

are isomorphic locally for the Zariski topology on X.

In other words, for any point $x \in X$ there is a regular function $g \in \mathbb{C}[X]$ such that $g(x) \neq 0$ and quadratic spaces P and Q are isomorphic as quadratic spaces over $\mathbb{C}[X]_g$.

A comment

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It is pretty clear now that one can try to state a rather general theorem in terms of principal $\mathbf{G}$-bundles. To do that recall the notion of a

PRINCIPAL $\mathbf{G}$-bundle

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A principal $\mathbf{G}$-bundle over X is a pair $(E, \nu : \mathbf{G} \times E \to E)$ above such that the map $p : E \to X$ is smooth surjective and

- the regular map $\mathbf{G} \times E \to E \times_X E$ taking (g, e) to $(g \cdot e, e)$ is an isomorphism of algebraic varieties;

In this case there exists a cover $X = \bigcup V_i$ in the complex topology on X and holomorphic isomorphisms $\varphi_i : \mathbf{G} \times V_i \to E|_{V_i} := p^{-1}(V_i)$ respecting as the projections onto V_i so the $\mathbf{G}$-actions on both sides.

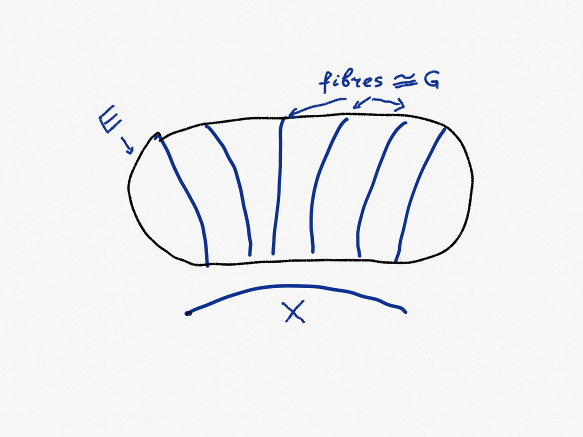
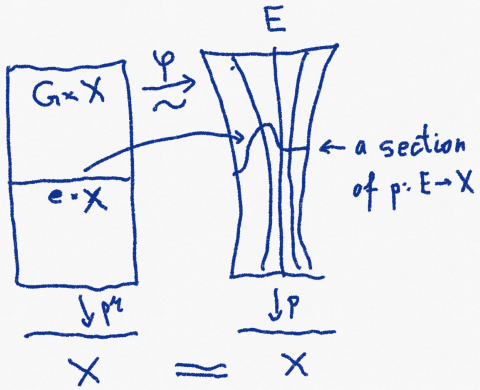


Рис.:

An isomorphism between principal G-bundles (E_1, ν_1) and (E_2, ν_2) is a morphism $\psi : E_1 \to E_2$ respecting the projections on X, and the G-actions.

A trivial G-bundle is a G-bundle isomorphic to G-bundle of the form $(\mathbf{G} \times X, \mu)$, where $g' \cdot (g, x) = ((g' \cdot g), x)$. A trivial bundle has a section. If a bundle E has a section s, then it is trivial. Indeed, the map $(g, x) \mapsto g \cdot s(x)$ identifies $\mathbf{G} \times X$ with E.

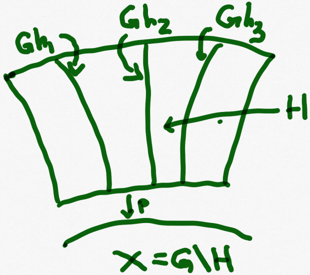


Many examples of principal $\mathbf{G}$-bundles are obtained by the following simple construction. Consider a closed embedding of algebraic groups $\mathbf{G} \subset \mathbf{H}$ and set $X = \mathbf{G}\backslash\mathbf{H}$ (the orbit variety of right cosets with respect to $\mathbf{G}$). Then the pair

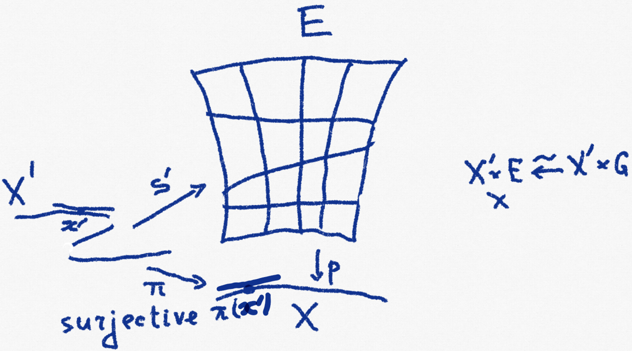
$$(\mathbf{H}, \nu : \mathbf{G} \times \mathbf{H} \to \mathbf{H}),$$

where ν takes (g, h) to $g \cdot h$ is a principal $\mathbf{G}$-bundle over X.

The fibres of the projection $p : \mathbf{H} \to X$ are right cosets of $\mathbf{H}$ with respect to the subgroup $\mathbf{G}$.



A principal G-bundle E over X is not necessary trivial locally for the Zariski topology on X. However it is always trivial locally for the étale topology on X. In a picture the latter means the following: here X' is smooth, $\pi : X' \to X$ is surjective and any point $x' \in X'$ one has $T_{X',x'} \cong T_{X,\pi(x')}$.



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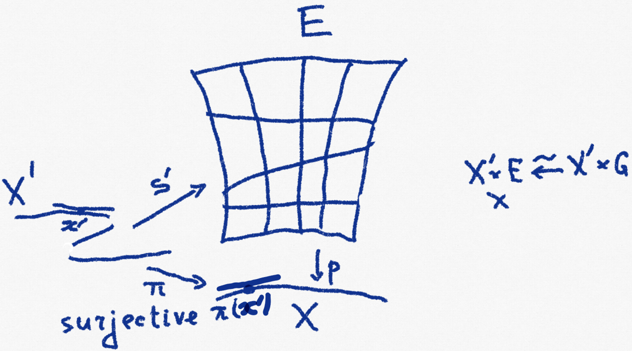

Examples of simple, semi-simple, reductive complex algebraic groups.

A reductive group is connected by an agreement due to Demazure and Grothendieck.
SL_n, PGL_n, SO_n, $Spin_n$, PGO_n^+, Sp_{2n}, PSp_{2n},
G_2, F_4, E_6, E_7, E_8,
$SL_3 \times E_6$, $Sp_{2n} \times Spin_m$
GL_n and GO_n, GSp_{2n} (the groups of similitudes).

We are ready now to state a very general result concerning principal G-bundles and extending the results from the introduction.

Theorem (R.Fedorov,I.Panin; 2013)

Let $\mathbf{G}$ be a simple (or a semi-simple, or even a reductive) complex algebraic group. Let X be an affine complex algebraic variety, smooth and irreducible and let E_1, E_2 be two principal $\mathbf{G}$-bundles over X. Suppose there is a non-zero regular function $f \in \mathbb{C}[X]$ such that the principal $\mathbf{G}$-bundles $E_1|_{X_f}$ and $E_2|_{X_f}$ are isomorphic over X_f.

Then the principal $\mathbf{G}$-bundles E_1 and E_2 are isomorphic locally for the Zariski topology on X.

Remark. Particularly, if E_1 is trivial over a non-empty Zariski open subset of X, then E_1 is trivial locally for the Zariski topology on X.

Examples illustrating the Theorem.

• **Let A_1 and A_2 be two algebras as in the Serre's theorem above.** *They are called Azumaya $\mathbb{C}[X]$-algebras.* **Suppose for a non-zero function $f \in \mathbb{C}[X]$ the $\mathbb{C}[X_f]$-algebras $(A_1)_f$ and $(A_2)_f$ are isomorphic. Then the $\mathbb{C}[X]$-algebras A_1 and A_2 are isomorphic locally for the Zariski topology on X.**

• **Let P and Q be the quadratic spaces over $\mathbb{C}[X]$ as in Ojanguren's theorem. Suppose they are in the same similarity class over the field $\mathbb{C}(X)$, then they are in the same similarity class locally for the Zariski topology on X.**

Non-constant case of the conjecture for complex algebraic varieties.

- Example 1. Let $a, b \in \mathbb{C}[X]^{\times}$. Consider an equation

$$T_1^2 - aT_2^2 = b \qquad (1)$$

If this equation has a solution over the field $\mathbb{C}(X)$ then for any point $x \in X$ there is a function $g \in \mathbb{C}[X]$ such that $g(x) \neq 0$ and the equation (1) has a solution in $\mathbb{C}[X_g]$.

- Example 2. Let $a, b, c \in \mathbb{C}[X]^{\times}$. Consider an equation

$$T_1^2 - aT_2^2 - bT_3^2 + abT_4^2 = c \qquad (2)$$

Suppose this equation has a solution over the field $\mathbb{C}(X)$. Then for any point $x \in X$ there is a function $g \in \mathbb{C}[X]$ such that $g(x) \neq 0$ and the equation (2) has a solution in $\mathbb{C}[X_g]$.

Reformulate these statements in terms of principal G-bundles for reductive group schemes over our complex algebraic variety X.

Recall for that notion of a reductive group X-scheme and a principal G-bundle.

Let X be as above. A smooth X-group scheme consists of the data $p : \mathbf{G} \to X, \mu : \mathbf{G} \times_X \mathbf{G} \to \mathbf{G}, i : \mathbf{G} \to \mathbf{G}, e : X \to \mathbf{G}$, where p, μ, i, e are regular maps. The requirements are the obvious ones.

• In the example (1) consider an X-group scheme defined by the equation $T_1^2 - aT_2^2 = 1$. Call it $\mathbf{T}$.

• In the example (2) consider an X-group scheme defined by the equation $T_1^2 - aT_2^2 - bT_3^2 + abT_4^2 = 1$. Call it $SL_{1,A}$, where A is the generalized quaternion $\mathbb{C}[X]$-algebra for the pair a, b.
One has $\mathbf{T} \cong \mathbb{C}^{\times} \times X$, $SL_{1,A} \cong SL_2(\mathbb{C}) \times X$, **locally for the complex topology on** X.

The following well-known definition shows that the two X-group schemes $\mathbf{T}$ and $SL_{1,A}$ are REDUCTIVE X-GROUP SCHEMES.

Being a bit non-precise, an X-group scheme $\mathbf{G}$ is called a reductive if for a complex algebraic reductive group $\mathbf{G_0}$

$$\mathbf{G} \cong \mathbf{G_0} \times \mathbf{X}$$

holomorphically isomorphic locally for the complex topology on X. Recall that $\mathbf{G}_0$ is required to be connected. The class of reductive group schemes contains the class of semi-simple group schemes which in turn contains the class of simple group schemes.

Examples: $\mathbf{T}$, $SL_{1,A}$, PGL_n, $Spin_Q$, G_2, F_4, E_6,E_7,E_8.

Let $\mathbf{G}$ be a reductive X-group scheme. *A principal* $\mathbf{G}$*-bundle over* X consists of data $(p: E \to X, \nu: \mathbf{G} \times_X E \to E)$ such that p is a smooth surjective regular map, ν is a $\mathbf{G}$-action respecting the projections on X and
1) the regular map $\mathbf{G} \times_X E \to E \times_X E$ taking (g, e) to (ge, e) is an isomorphism of algebraic varieties;

A principal $\mathbf{G}$-bundle E *is called trivial* if there is an isomorphism $E \to \mathbf{G}$ over X, which respects the obvious left $\mathbf{G}$-action on both sides. E *is trivial* if and only if there is a section $s: X \to E$ of the projection $p: E \to X$.

The equation (1) above defines a principal $\mathbf{T}$-bundle.
The equation (2) above defines a principal $SL_{1,A}$-bundle.

Theorem non-constant case: R.Fedorov,I.Panin;2013

Let $\mathbf{G}$ be a complex algebraic reductive X-group scheme and E be a principal $\mathbf{G}$-bundle. Suppose for a non-zero function f the principal $\mathbf{G}$-bundle $E|_{X_f}$ is trivial over X_f. Then E is trivial locally for the Zariski topology on X.

Corollary

Let $\mathbf{H}$, $\mu : \mathbf{H} \to \mathbb{G}_{m,X}$ and $\lambda \in \mathbb{C}[X]^{\times}$. Suppose the kernel $ker(\mu)$ is a reductive X-group scheme. If the equation $\mu(h) = \lambda$ has a solution over $\mathbb{C}(X)$, then it has a solution locally for the Zariski topology on X.

General case of the conjecture

Let $U = \text{Spec}(R)$ be an irreducible regular scheme and $\mathbf{G}$ be a reductive U-group scheme. Recall that a U-scheme E with an action of $\mathbf{G}$ is called a principal $\mathbf{G}$-bundle over U, if E is smooth and surjective over U and the morphism $\mathbf{G} \times_U E \to E \times_U E$ taking (g, e) to (ge, e) is an isomorphism (see [Gro5, Section 6]).

Conjecture[Serre (1958), Grothendieck (1968)]. Let K be the fraction field of a regular local ring R. If $E(K) \neq \emptyset$, then $E(R) \neq \emptyset$.

Theorem. If R is a regular local ring **containing a field**, then the above conjecture holds. That is $[E(K) \neq \emptyset \Rightarrow E(R) \neq \emptyset]$.

This theorem is proved by R.Fedorov and the author in [FP, 2013] in the case, when R contains an infinite field. It is proved by the author in [Pan, 2015], when R contains a finite field.

Corollary. Let R be a regular local ring, K be its field of fractions, $U = \text{Spec}(R)$. Let $\mu : \mathbf{H} \to \mathbb{G}_{m,U}$ be a smooth U-group morphism, where $\mathbf{H}$ is a reductive U-group scheme. Suppose the kernel $ker(\mu)$ is a reductive U-group scheme. Then the inclusion of R into K induces an injection

$$R^{\times}/\mu(\mathbf{H}(R)) \hookrightarrow K^{\times}/\mu(\mathbf{H}(K)).$$

History of the topic

History of the topic. — In his 1958 paper Jean-Pierre Serre asked whether a principal bundle is Zariski locally trivial, once it has a rational section (see [Ser, Remarque, p. 31]). In his setup the group is any algebraic group over an algebraically closed field. He gave an affirmative answer to the question when the group is PGL(n) (see [Ser, Prop. 18]) and when the group is an abelian variety (see [Ser, Lemme 4]). In the same year, Alexander Grothendieck asked a similar question (see[Gro1,Remarque 3, pp. 26–27]). A few years later, Grothendieck conjectured that the statement is true for any semi-simple group scheme over any regular local scheme (see[Gro 4,Remarque 1.11.a]). Now by the Grothendieck–Serre conjecture we mean Conjecture 1 though this may be slightly inaccurate from historical perspective. Many results corroborating the conjecture are known.

Here is a list of known results in the same vein, corroborating the Grothendieck–Serre conjecture.

- The case when the group is PGL_n and the base field is algebraically closed is done by J.-P.Serre in 1958
- The case when the group scheme is PGL_n and the ring R is an arbitrary regular local ring is done by A.Grothendieck in 1968
- The case when the local ring R contains a field of characteristic not 2 the group is SO_n over the ground field is done by M.Ojanguren in 1982
- The case of an arbitrary reductive group scheme over a discrete valuation ring or over a henselian ring is solved by Y. Nisnevich in 1984
- The case, where $\mathbf{G}$ is an arbitrary torus over a regular local ring, was settled by J.-L. Colliot-Thélène and J.-J. Sansuc in 1987

• The case, when $\mathbf{G}$ is quasi-split reductive group scheme over arbitrary two-dimensional local rings, is solved by Y. Nisnevich in 1989

• The case, where the group scheme $\mathbf{G}$ comes from an infinite perfect ground field, solved by J.-L. Colliot-Thélène, M. Ojanguren in 1992 As far as we know this work was inspired by the one [Oj1,1982].

• The case, where the group scheme $\mathbf{G}$ comes from an arbitrary infinite ground field, solved by M. S. Raghunatan 1994

• O. Gabber announced in 1994 a proof for group schemes coming from arbitrary ground fields (including finite fields).

• For the group scheme $\mathsf{SL}_{1,A}$, where A is an Azumaya R-algebra and R contains a field the conjecture is solved by A.Suslin and the author in 1998

• For the unitary group scheme $\mathsf{U}^{\epsilon}_{A,\sigma}$, where (A,σ) is an Azumaya R-algebra with involution R contains a field of characteristic not 2 the conjecture is solved by M.Ojanguren and the author in 2001

• For the special unitary group scheme $\mathsf{SU}_{A,\sigma}$, where (A,σ) is an Azumaya R-algebra with a unitary involution and R contains a field of characteristic not 2 the conjecture is solved by K. Zainoulline in 2001

• For the spinor group scheme Spin_Q of a quadratic space Q over R containing a field of characteristic not 2 the conjecture is solved M. Ojanguren, K. Zainoulline and the author in 2004

• Under an isotropy condition on $\mathbf{G}$ the conjecture is proved by A.Stavrova, N.Vavilov and the author in a series of preprints in 2009, published as papers in 2015 and in 2016

• The case of strongly inner simple adjoint group schemes of the types E_6 and E_7 is done by the second author, V. Petrov, A. Stavrova and the second author in 2009. No isotropy condition is imposed there.

• The case, when $\mathbf{G}$ is of the type F_4 with trivial f_3-invariant and the field is infinite and perfect, is settled by V. Petrov and A. Stavrova in 2009

• The case, when $\mathbf{G}$ is of the type F_4 with trivial g_3-invariant and the field is of characteristic zero, is settled by V. Chernousov in 2010

• The conjecture is solved when R contains an infinite field, by R.Fedorov and the author in a preprint in 2013 and published in 2015.

• The conjecture is solved by the author in the case, when R contains a finite field in 2015 (for a better structured proof see [Pan3,2017]).

So, the conjecture is solved in the case, when R contains a field.

The case of mixed characteristic is widely open. Let us indicate two recent interesing preprints [F1] and [PS3]. In [F1] the conjecture is solved for a large class of regular local rings of mixed characteristic assuming that $\mathbf{G}$ splits. In [PS3] the conjecture is solved for any semi-local Dedekind domain providing that $\mathbf{G}$ is simple simply-connected and $\mathbf{G}$ contains a torus $\mathbb{G}_{m,R}$.

A sketch of the proof in the constant simply connected case assuming the base field is $\mathbb{C}$. Suppose a principal $\mathbf{G}$-bundle E over X is trivial over X_f. Then there exists a principal $\mathbf{G}$-bundle over $\mathbb{C} \times U$ as on the picture. One has isomorphisms $\mathbf{G} \times U \cong E_t|_{1 \times U} \cong E_t|_{0 \times U} = E|_U$.