Finite groups without elements of order $2p$ for an odd prime p

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Abstract. We consider the problem of desription of finite groups without elements of order $2p$ for an odd prime p. A rich history of the study of particular cases of the problem is given. As main result, we prove new general structural theorem on finite non-solvable groups without elements of order $2p$ for an odd prime p. The theorem reduces largely a solving the problem to investigating almost simple groups and faithful 2-modular representations of quasisimple-groups. The theorem reinforces essentially well-known Vasil'ev-Gorshkov theorem and can be applied to obtaining new arithmetical characterizations of finite groups.

Our notation and terminology are mostly standard and can be found in [1, 2].

Let G be a finite group. Denote by $\pi(G)$ the set of all prime divisors of the order of G. The Gruenberg–Kegel graph, or the prime graph, of G is the graph $\Gamma(G)$ with vertex set $\pi(G)$ in which two vertices p and q are adjacent if and only if $p \neq q$ and G has an element of order pq. A set of a pairwise non-adjacent vertices of a graph is called its coclique. Denote by $t(G)$ the greatest cardinality of cocliques of $\Gamma(G)$ and by $t(2, G)$ the greatest cardinality of cocliques containing 2 of $\Gamma(G)$. A finite group X is almost simple whenever $S \leq X \leq Aut(S)$ for some finite nonabelian simple group S; equivalently, provided that the socle of X is a finite nonabelian simple group. A finite group X is quasisimple whenever $X' = X$ and $X/Z(X)$ is a finite nonabelian simple group.

It is well known the dominating role of involutions in the thery of finite non-solvable groups and especially in the classification of finite simple groups (CFSG). If G is a finite group of even order and $2 \neq p \in \pi(G)$ then it is important to know whether the vertices 2 and p are adjacent in the graph $\Gamma(G)$. We shall call an W_p-group for an odd prime p a finite group of order dividing by $2p$ and having no elements of order $2p$; the vertices 2 and p in the Gruenberg–Kegel graph of such group are non-adjacent. Denote by W the class of all W_p-groups when p runs all odd primes. Then W coincides with the class of finite groups G such that $t(2, G) \geq 2$.

Any finite simple group belongs to the class W, except the alternating group of some degree. Finite groups with disconnected Gruenberg-Kegel graphs belong also to W. Thus, the class W is wide and the following problem arises naturally.

Problem. *Describe W_p-groups for an odd prime p, at least small ones.*

Given a finite group G and an odd prime p, denote by $W_p(G)$ the group $O^{\{2,p\}'}(G/O_{\{2,p\}'}(G))$ It is clear that G is a W_p-group if and only if $W_p(G)$ is a W_p-group.

The history of results on particular cases of Problem is very rich.

Burnside (1900) considered the case when the order of any element of a finite group either is odd or equals 2.

G. Higman (1957) described finite groups whose element orders are prime powers (Shi called such groups shortly as $EPPO$-groups). It is clear that the connected components of the Gruenberg-Kegel graph of a $EPPO$-group are one-element.

A finite group of order dividing by a prime p is called a C_{pp}-group if the centralizer of any its non-trivial element is a p-group. It is clear that a $EPPO$-group G is C_{pp}-group for any $p \in \pi(G)$.

Back at the dawn of classification finite simple groups (CFSG), M. Suzuki (1961-1962) in his pioneer fundamental papers obtained a description of C_{22}-groups, i. e., groups which are W_p-groups for all odd prime divisors p of their orders. In the sequal, this description was refined and reduced to a criterion in works of G. Higman (1968), P. Martino (1972), Stewart (1973) and Brandl (1981). As a corollary, a complete description of $EPPO$-groups was obtained.

A complete description of non-primary C_{33}-groups is obtain by G. Higman (1968), Stewart (1973) and Fletcher, Stellmacher, and Stewart (1977).

A description of non-primary C_{55}-groups is obtained in the papers by Dolphi, Jabara, Lucido (2004) and Astill, C. Parker and Waldecker (2012).

It is remained open the problem of describing C_{pp}-groups for a prime $p > 5$.

It is clear that the Gruenberg–Kegel graph of any non-primary C_{pp}-group is disconnected.

The first result about finite groups with disconnected Gruenberg-Kegel graph is the well-known structural theorem (Gruenberg-Kegel Theorem) obtained by Gruenberg and Kegel about 1975 in an unpublished paper. The proof of this theorem was published in the paper of Williams (1981), post-graduate of Gruenberg.

Williams (1981) obtained also an explicit description of connected components of the Gruenberg-Kegel graph for all finite simple non-abelian groups except the groups of Lie type of even characteristic.

In 1989, AK [4] obtained such description for the remaining case of the groups of Lie type of even characteristic. Later in 1993, this result was repeated by Iiyori and Yamaki. But later, some inaccuracies in all three papers were found. In a joint work of AK and Mazurov [5], the corresponding tables were corrected.

The classification of connected components of Gruenberg-Kegel graph for finite simple groups were applied by Lucido (1999, 2002) for obtaining analogous classification for all finite almost simple groups.

It is remained open the natural problem of describing the finite non-solvable groups with disconnected Gruenberg-Kegel graph, which are not almost simple.

In 2005, A.V. Vasil'ev observed that the proof of the Gruenberg-Kegel Theorem uses essentially the fact that a finite group G with disconnected Gruenberg–Kegel graph contains an element of odd prime order which does not adjanced in Γ to 2. A.V. Vasil'ev in [7] proved a wide generalization of the Gruenberg-Kegel Theorem for non-solvable groups. This result was a few sharpened by A.V. Vasil'ev and Gorshkov in [8]. Vasil'ev-Gorshkov Theorem can be considered as a general structural theorem for non-solvable groups from class W with an emphasis on a relation with their Gruenberg-–Kegel graphs.

Now we consider our results on Problem.

The non-abelian finite simple W_3-groups were determined in 1977 in the three independent articles of Podufalov, Fletcher, Stellmacher, and Stewart, as well as Gordon. The problem of describing general finite W_3-groups remained open for more than 40 years before In 2018, AK and Minigulov in [6] solved it without using CFSG.

Recently in [3], we solved Problem for solvable W_p-groups and $p > 3$. Now using this result, we reinforce Vasil'ev-Gorshkov Theorem with an emphasis on normal structure of investigated groups proving the following main theorem.

Theorem. *Let G be a non-solvable W_p-group for an odd prime p, $K = S(G)$ and $\overline{G} = G/K$. Then $\overline{G}$ is an almost simple group with the socle S, p does not divide the index $|\overline{G} : S|$, $t(S) \geq t(G) - 1$, $t(2, S) \geq t(2, G)$, and one of the following assertions holds:*

(1) *$K = O_{\{2,p\}'}(G))$;*

(2) *p divides $|K|$, a Sylow 2-subgroup of G is (generalized) quaternion group, $G/O(G)$ is isomorphic to either $2{\cdot}A_7$ or an extension of $SL_2(q)$ for odd $q > 3$ by a cyclic group of either odd order or a doubled odd order, $O_{p',p,p'}(O(G)) = O(G)$, an involution from $K = Z^*(G)$ inverts some (abelian) Sylow p-subgroup of $O(G)$ and centralizes $O(G)/O_{p',p}(O(G))$, $t(G) = t(2, S) = 3$, and $t(2, G) = 2$;*

(3) *p does not divide $|K|$, 2 divides $|K|$, a Sylow p-subgroup of G is cyclic, $O_{2',2,2'}(K) = K$ and S centralizes $K/O_{2',2}(K)$.*

Remark. In the case (3) of Theorem, the last term E of the derived series of $G/O_{2',2}(K)$ is a quasisimple W_p-group such that $EK/O_{2',2}(K)$ is a central product of $K/O_{2',2}(K)$ and E, $E/Z(E)$ is isonorphic to S, and E acts faithfully on $O_{2',2}(K)/O(K)$.

Therefore, Theorem reduces largely a solving Problem to investigating the cases (1) and (3) of Theorem, i. e., almost simple W_p-groups and faithful 2-modular representations of quasisimple W_p-groups.

Recently, we determined in [3] all almost simple W_5-groups.

The results we have obtained can be applied to obtaining new arithmetical characterizations of finite groups.

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