Classical versus Tropical Algebra

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Tropical semi-ring

Tropical semi-ring T is endowed with operations $\oplus$, $\otimes$.
If T is an ordered semi-group then T is a tropical semi-ring with inherited operations $\oplus := \min$, $\otimes := +$.
If T is an ordered (resp. abelian) group then T is a *tropical semi-skew-field* (resp. *tropical semi-field*) w.r.t. $\oslash := -$.
Examples • $\mathbb{Z}^+ := \{0 \le a \in \mathbb{Z}\}$, $\mathbb{Z}^+_\infty := \mathbb{Z}^+ \cup \{\infty\}$ are commutative tropical semi-rings. ∞ plays a role of 0, in its turn 0 plays a role of 1;
• $\mathbb{Z}$, $\mathbb{Z}_\infty$ are semi-fields;
• $n \times n$ matrices over $\mathbb{Z}_\infty$ form a non-commutative tropical semi-ring: $(a_{ij}) \otimes (b_{kl}) := (\oplus_{1 \le j \le n} a_{ij} \otimes b_{jl})$.

Tropical polynomials

Tropical monomial $x^{\otimes i} := x \otimes \cdots \otimes x$, $Q = a \otimes x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n}$, its *tropical degree* $\mathrm{trdeg} = i_1 + \cdots + i_n$. Then $Q = a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n$.
Tropical polynomial $f = \bigoplus_j (a_j \otimes x_1^{i_{j1}} \otimes \cdots \otimes x_n^{i_{jn}}) = \min_j\{Q_j\}$;
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Historical sources of the tropical algebra

Logarithmic scaling of the reals (mathematical physics)

Define two operations on positive reals, replacing addition and multiplication:

$a, b \to t \cdot \log(\exp(a/t) + \exp(b/t)), \quad \lim_{t \to 0} = \max\{a, b\}$

$a, b \to t \cdot \log(\exp(a/t) \cdot \exp(b/t)) = a + b$

Thus, the ”dequantization” of the logarithmic scaling is a tropical semi-ring

Solving systems of polynomial equations in Puiseux series (algebraic geometry)

The field of Puiseux series $F((t^{1/\infty})) \ni a_0 \cdot t^{i/q} + a_1 \cdot t^{(i+1)/q} + \cdots, 0 < q \in \mathbb{Z}$ over an algebraically closed field F is algebraically closed. In the (Newton) algorithm for solving a system of polynomial equations $f_i(X_1, \dots, X_n) = 0, 1 \le i \le k$ with $f_i \in F((t^{1/\infty}))[X_1, \dots, X_n]$ in Puiseux series the leading exponents i_j/q_j in $X_j = a_{0j} \cdot t^{i_j/q_j} + \cdots$ satisfy a tropical polynomial system (due to cancelation of the leading terms).

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Minimal weights of paths in a graph (computer science)

For a graph with weights w_{ij} on edges (i, j) for any k to compute for each pair of vertices i, j the minimal weight of paths of length k between i and j. This is equivalent to computing the tropical k-th power of matrix (w_{ij}).

Scheduling

Let several jobs i should be executed by means of several machines j with times of execution t_{ij}. The restrictions like that job i_0 should be executed after job i are imposed. Denoting by unknown x_{ij} a starting moment of execution of i by j, the latter restriction is expressed as $x_{i_0,j_0} \geq \min_j\{x_{ij} + t_{ij}\}$. Another sort of restrictions is that a machine can't execute two jobs simultaneously, i. e. $x_{i_1,j} \geq x_{ij} + t_{ij}$. It leads to a system of min-plus linear inequalities, the problem being equivalent to tropical linear systems.

This approach is employed in scheduling of Dutch and Korean railways.

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Some other applications of tropical algebra

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If T is an ordered semi-group then tropical linear function over T can be written as $\min_{1\le i\le n}\{a_i + x_i\}$.

Tropical linear system

$$\min_{1\le j\le n}\{a_{i,j} + x_j\},\ 1 \le i \le m$$

(or $(m \times n)$-matrix $A = (a_{i,j})$) has a *tropical solution* $x = (x_1 \dots, x_n)$ if for every row $1 \le i \le m$ there are two columns $1 \le k < l \le n$ such that

$$a_{i,k} + x_k = a_{i,l} + x_l = \min_{1\le j\le n}\{a_{i,j} + x_j\}$$

Coefficients $a_{i,j} \in \mathbb{Z}_\infty := \mathbb{Z} \cup \{\infty\}$. Not all $x_j = \infty$. For $a_{i,j} \in \mathbb{Z}$ we assume $0 \le a_{i,j} \le M$.

$n \times n$ matrix $(a_{i,j})$ is **tropically non-singular** if $\min_{\pi \in S_n}\{a_{1,\pi(1)} + \cdots + a_{n,\pi(n)}\}$ is attained for a *unique* permutation π

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One can solve an $m \times n$ tropical linear system A within complexity polynomial in n, m, M. (Akian-Gaubert-Guterman; G.)
Moreover, the algorithm either finds a solution over $\mathbb{Z}_\infty$ or produces an $n \times n$ tropically nonsingular submatrix of A.

Corollary

The problem of solvability of tropical linear systems is in the complexity class NP $\cap$ coNP.

Remark

- *My algorithm has also a complexity bound polynomial in $2^{nm}, \log M$ (as well as an obvious algorithm which invokes linear programming);*
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The theorem on complexity of solving tropical linear systems implies

Corollary

The following statements are equivalent

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- *The corollary holds for matrices over $\mathbb{R}_\infty$.*
- *For matrices A with* **finite** *coefficients from $\mathbb{R}$ it was proved by Develin-Santos-Sturmfels.*
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The theorem on complexity of solving tropical linear systems implies

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The following statements are equivalent

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Theorem

Solvability of tropical polynomial systems is NP-complete (Theobald)

How to reduce tropical polynomial systems to tropical linear ones?

In the classical algebra for this aim serves Nullstellensatz.

In the tropical world the direct version of Nullstellensatz is false even for linear univariate polynomials: $X \oplus 0$, $X \oplus 1$ do not have a tropical solution, while their (tropical) ideal does not contain 0 or any other monomial (tropical monomials are the only polynomials without tropical zeroes).

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"Dual" (classical) Nullstellensatz

For polynomials $g_1, \ldots, g_k \in \mathbb{C}[X_1, \ldots, X_n]$ consider an infinite Macaulay matrix C with the columns indexed by monomials X^I and the rows by shifts $X^J \cdot g_i$ with their coefficients being entries of C.

Nullstellensatz: system $g_1 = \cdots = g_k = 0$ has no solution iff a linear combination of the rows of a suitable *finite* submatrix C_N of C (generated by a set of rows $X^J \cdot g_i$, $1 \leq i \leq k$ of C with degrees of monomials $|J| \leq N$) equals vector $(1, 0, \ldots, 0)$.

Effective Nullstellensatz: $N \leq (\max_{1 \leq i \leq k}\{\deg(g_i)\})^{O(n)}$.
(**Galligo, Heintz, Giusti; Kollar**)

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Classical homogeneous (projective) effective Nullstellensatz

Let $g_0, \dots, g_k \in \mathbb{C}[X_0, \dots, X_n]$ be homogeneous polynomials with $\deg(g_0) \geq \deg(g_1) \geq \cdots$.

Theorem

*System $g_0 = \cdots = g_k = 0$ has a solution in the projective space iff the ideal generated by $g_0, \dots, g_k$ does not contain the power $(X_0, \dots, X_n)^{N_0}$ of the coordinate ideal for $N_0 = \deg(g_0) + \cdots + \deg(g_n) - n$. (**Lazard**)*

In the dual form this means that system $g_0 = \cdots = g_k = 0$ has a solution in the projective space iff the homogeneous linear system with submatrix $C_{N_0}^{(hom)}$ of the Macaulay matrix C generated by the columns with the degrees of monomials equal N_0, has a non-zero solution.

Thus, the bound on the degrees of monomials in the Macaulay matrix in the affine Nullstellensatz is roughly the product of the degrees (Bezout number) of the polynomials in the system, while the bound in the projective Nullstellensatz is roughly the sum of the degrees.

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Tropical dual effective Nullstellensatz: finite case

Assume w.l.o.g. that for tropical polynomials $h = \bigoplus_J (a_J \otimes X^{\otimes J})$ in n variables which we consider, function $J \to a_J$ is concave on $\mathbb{R}^n$. This assumption does not change tropical prevarieties, the results hold without it, but it makes the geometric intuition more transparent.

For tropical polynomials $h_1, \dots, h_k$ consider (infinite) Macaulay matrix H with the rows indexed by $X^{\otimes I} \otimes h_i$ for $I \in \mathbb{Z}^n$, $1 \leq i \leq k$.

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*Tropical polynomials $h_1, \dots, h_k$ have a solution over $\mathbb{R}$ iff tropical linear system $H_N \otimes (z_0, \dots, z_L)$ has a solution over $\mathbb{R}$ where H_N is (finite) submatrix of H generated by its rows $X^{\otimes I} \otimes h_i$, $1 \leq i \leq k$ for $|I| \leq N = (n+2) \cdot (\mathrm{trdeg}(h_1) + \cdots + \mathrm{trdeg}(h_k))$. (**G.-Podolskii**)*

Conjecture is that the latter bound is $O(trdeg(h_1) + \cdots + trdeg(h_k))$.

In case $k = 2, n = 1$ the bound $trdeg(h_1) + trdeg(h_2)$ was proved by **Tabera** using the classical resultant and **Kapranov's** theorem: for a polynomial $f \in R((t^{1/\infty}))[x_1, \dots, x_n]$ it holds:

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*Tropical polynomials $h_1, \ldots, h_k$ have a solution over $\mathbb{R}$ iff tropical linear system $H_N \otimes (z_0, \ldots, z_L)$ has a solution over $\mathbb{R}$ where H_N is (finite) submatrix of H generated by its rows $X^{\otimes I} \otimes h_i$, $1 \le i \le k$ for $|I| \le N = (n+2) \cdot (\mathrm{trdeg}(h_1) + \cdots + \mathrm{trdeg}(h_k))$. (**G.-Podolskii**)*

Conjecture is that the latter bound is $O(trdeg(h_1) + \cdots + trdeg(h_k))$.

In case $k = 2, n = 1$ the bound $trdeg(h_1) + trdeg(h_2)$ was proved by **Tabera** using the classical resultant and **Kapranov's** theorem: for a polynomial $f \in R((t^{1/\infty}))[x_1, \ldots, x_n]$ it holds:

$Prevariety(Trop(f)) = Trop(Variety(f))$

Tropical dual effective Nullstellensatz: finite case

Assume w.l.o.g. that for tropical polynomials $h = \bigoplus_J (a_J \otimes X^{\otimes J})$ in n variables which we consider, function $J \to a_J$ is concave on $\mathbb{R}^n$. This assumption does not change tropical prevarieties, the results hold without it, but it makes the geometric intuition more transparent.
For tropical polynomials $h_1, \dots, h_k$ consider (infinite) Macaulay matrix H with the rows indexed by $X^{\otimes I} \otimes h_i$ for $I \in \mathbb{Z}^n$, $1 \le i \le k$.

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(Convex)-geometrical rephrasing of the tropical dual Nullstellensatz over $\mathbb{R}$ (finite case)

For a tropical polynomial $h = \bigoplus_J (a_J \otimes X^{\otimes J})$ consider its extended Newton polyhedron G being the convex hull of the graph $\{(J, a) : a \leq -a_J\} \subset \mathbb{R}^{n+1}$. As vertices of G consider all the points of the form (I, c), $I \in \mathbb{Z}^n$ on the boundary of G. Let G_i correspond to h_i, $1 \leq i \leq k$. Denote by $G^{(I)} := G + (I, 0)$ a horizontal shift of G. Solution $Y := \{(J, y_J)\} \subset \mathbb{Z}^n \times \mathbb{R}$ of a tropical linear system $H \otimes Y$ treat also as a graph on $\mathbb{Z}^n$.

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For any I, i take the maximal $b := b_{I,i}$ such that a vertical shift $G_i^{(I)} + (0, b) \leq Y$ (pointwise as graphs on $\mathbb{Z}^n$).

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Tropical dual effective Nullstellensatz over $\mathbb{R}_\infty$

Theorem

*A system of tropical polynomials $h_1, \dots, h_k$ has a zero over $\mathbb{R}_\infty$ iff the tropical non-homogeneous linear system with a finite submatrix H_N of the Macaulay matrix H generated by its rows $X^{\otimes I} \otimes h_i$, $1 \le i \le k$ has a tropical solution over $\mathbb{R}_\infty$ where tropical degrees $|I| < N = O(kn^2(2\max_{1\le j\le k}\{\mathrm{trdeg}(h_j)\})^{O(\min\{n,k\})})$ (**G.-Podolskii**)*

Thus, the following table of bounds for effective Nullstellensätze demonstrates a similarity of tropical geometry with the complex one

Classical	Projective	Affine
Tropical	Finite ($\mathbb{R}$)	Infinite ($\mathbb{R}_\infty$)
Bound	Sum of degrees	Product of degrees

What is the reason of this analogy between projective vs. affine and finite vs. infinite?

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Tropical Varieties and Prevarieties

$K = \mathbb{C}((t^{1/\infty})) = \{c = c_0 t^{i_0/q} + c_1 t^{(i_0+1)/q} + \cdots\}$

is a field of Puiseux series where $i_0 \in \mathbb{Z}$, $1 \leq q \in \mathbb{Z}$.

Consider an ideal $I \subset K[X_1, \dots, X_n]$, the variety of its solutions $U(I) \subset K^n$.

Tropicalization $Trop(c) = i_0/q$, $Trop(0) = \infty$.

The closure in the Euclidean topology $V := \overline{Trop(U(I))} \subset \mathbb{R}^n$ is called the **tropical variety** of I.

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Bounds on Betti numbers of a tropical prevariety via the volume of Minkowski sum of Newton polytopes

Denote by $P_l \subset \mathbb{R}^n$ Newton polytope of f_i, $1 \leq i \leq k$.

Theorem

The number of faces of all dimensions of a tropical prevariety $V = V(f_1, \ldots, f_k)$ does not exceed $(2^{n+1} - 1) \cdot n! \cdot Vol_n(P_1 + \cdots + P_k)$.

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*(**Weak inequality of discrete Morse theory, R. Forman**). l-th Betti number (the rank of l-th homology group) of V is less or equal to the number of l-dimensional faces of V.*

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*(**G. - N. Vorobjov**). For* $\mathrm{trdeg}(f_i) \leq d,\ 1 \leq i \leq k$ *the sum of Betti numbers of* V *is less than* $(2^{n+1} - 1) \cdot (kd)^n$.

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For a tropical polynomial $f = \min_{1\leq j\leq m}\{a_j + \sum_{1\leq i\leq n} t_{j,i}X_i\}$ its shift

$$f_{s_1,\ldots,s_n} := \min_{1\leq j\leq m}\{a_j + \sum_{1\leq i\leq n}(t_{j,i}+s_i)X_i\} \in I(f).$$

Fix an integer N and introduce N^n variables $\{u(k_1,\ldots,k_n) : 0 \leq k_1,\ldots,k_n < N\}$. A **linearization** of $f_{s_1,\ldots,s_n}$ is a tropical linear polynomial $\min_{1\leq j\leq m}\{a_j + u(t_{j,1}+s_1,\ldots,t_{j,n}+s_n)\}$, provided that $0 \leq t_{j,1}+s_1,\ldots,t_{j,n}+s_n < N$.

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If $(x_1,\ldots,x_n) \in V(f)$ then point

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One can literally generalize the entropy $H(I)$ to semiring ideals I.

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The tropical prevariety U_N plays a role similar to the quotient ring $F[X_1, \ldots, X_n]/(g)$ with respect to the filtration degree N in classical algebra.

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i) For a tropical polynomial $f = \min\{2X, 1+X, 0\}$ its entropy $H(f) = 1/4$;
ii) for a tropical polynomial $f = \min\{sX, (s-1)X, \dots, X, 0\}$ its entropy $H(f) = 1 - 2/(s+1)$.

Sharp bounds on the entropy.

Theorem

(**N. Elizarov**).
i) If $H(f) > 0$ then $H(f) \geq 1/4$;
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Newton polygon of a tropical univariate polynomial

For a tropical polynomial $f = \min_{0 \leq k \leq s}\{a_k + kX\}$ its **Newton polygon** $P(f) \subset \mathbb{R}^2$ is the convex hull of the vertical rays $\{(k, y) : y \geq a_k\}, 0 \leq k \leq s$.

Theorem

The entropy $H(f) = 0$ iff all the points (k, a_k), $a_k < \infty$, $0 \leq k \leq s$ are the vertices of Newton polygon $P(f)$, and the indices k such that $a_k < \infty$ form an arthmetic progression.

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Tropical Hilbert function for univariate polynomials

For a tropical polynomial $f = \min_{0 \le k \le s}\{a_k + kX\}$ with finite coefficients $0 \le a_k \le m$, $0 \le k \le s$ consider a tropical linear prevariety $U_N(f) \subset \mathbb{R}^N$ of the points $(u_1, \dots, u_N)$ such that for each $1 \le j \le N - s$ the minimum in

$$\min_{0 \le k \le s}\{u_{j+k} + a_k\}$$

is attained at least twice. The tropical Hilbert function $T_N(f) = \dim(U_N(f))$.

Theorem

$T_N(f)$ is the sum of the linear function $H(f) \cdot N$ and a periodic function with an integer period (for $N > (ms)^{O(s)}$)*. The period does not exceed* $\exp((ms)^{O(s)})$*. In addition, the entropy $H(f)$ is a rational number.*

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Radical of a tropical prevariety

For a tropical prevariety $V \subset \mathbb{R}^n$ its **radical** $rad(V)$ is the semiring ideal of all the tropical polynomials for which points of V are tropical zeroes. For a semiring ideal I its radical is $rad(V(I))$.

Conjecture. For any semiring ideal I it holds $H(rad(I)) = 0$.

Strong conjecture. For a semiring ideal I of tropical polynomials in n variables its tropical Hilbert function $T_N(rad(I))$ is a polynomial of degree at most $n-1$ (for sufficiently large N).

Theorem

- *If V consists of a finite number of points then $H(rad(V)) = 0$;*
- *the strong conjecture holds for univariate polynomials;*
- *if $f = \min_{1 \le j \le r}\{t_{j,1}X + t_{j,2}Y\}$ is a tropical polynomial in 2 variables then $H(rad(f)) = 0$.*

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Consider a tropical quadratic polynomial $f = \min\{0, X, Y, X+Y\}$. Then $H(f) \geq 1/2$.

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