

On the Baer–Suzuki width

Danila Revin

Abstract. A class $\mathfrak{X}$ of finite groups is called radical if every finite group G has the largest normal subgroup $G_{\mathfrak{X}}$ belonging to $\mathfrak{X}$. According to the definition by Gordeev, Grunevald, Kunyavskii, and Plotkin, for a radical class $\mathfrak{X}$, the Baer–Suzuki width $\text{BS}(\mathfrak{X})$ is the smallest nonnegative integer m such that

$$G_{\mathfrak{X}} = \{x \in G \mid \langle x_1, \dots, x_m \rangle \in \mathfrak{X} \text{ for all } x_1, \dots, x_m \text{ conjugate to } x\}$$

for every finite group G , or, if no such m exists, we set $\text{BS}(\mathfrak{X}) = \infty$. A special case of a radical class is a class that is complete in the sense of Wielandt, i.e., a non-empty class that is closed under taking subgroups, homomorphic images, and extensions. Recently, the speaker proved that the Baer–Suzuki width of an arbitrary complete class is finite. Now, we present an almost unimprovable estimate of the Baer–Suzuki width of a complete class and give examples of classes with infinite Baer–Suzuki width.

Introduction

According to Wielandt [1], a nonempty class $\mathfrak{X}$ of finite groups is said to be *complete* if $\mathfrak{X}$ is closed under taking

- subgroups ($G \in \mathfrak{X} \Rightarrow H \in \mathfrak{X}$ for every $H \leq G$),
- homomorphic images ($G \in \mathfrak{X} \Rightarrow \phi(G) \in \mathfrak{X}$ for every group homomorphism ϕ with domain G), and
- extensions ($N \in \mathfrak{X}$ and $G/N \in \mathfrak{X} \Rightarrow G \in \mathfrak{X}$ for every group G and $N \trianglelefteq G$).

For a complete class $\mathfrak{X}$, each finite group G has the $\mathfrak{X}$ -*radical* $G_{\mathfrak{X}}$ —the largest normal subgroup which belongs to $\mathfrak{X}$.

The Baer–Suzuki theorem states that *if $\mathfrak{X}$ is the class of p -groups where p is a prime, then*

$$G_{\mathfrak{X}} = \{x \in G \mid \langle x_1, x_2 \rangle \in \mathfrak{X} \text{ for all } x_1, x_2 \text{ conjugate to } x\}$$

in every finite group G . Gordeev, Grunewald, Kunyavskii, and Plotkin [3] and independently Flavell, Guralnick, and Guest [2] proved that *if $\mathfrak{X}$ is the class of*

solvable groups, then

$$G_{\mathfrak{X}} = \{x \in G \mid \langle x_1, x_2, x_3, x_4 \rangle \in \mathfrak{X} \text{ for all } x_1, x_2, x_3, x_4 \text{ conjugate to } x\}.$$

In [4, Definition 1.15], Gordeev, Grunewald, Kunyavskii, and Plotkin introduced the following definition.

Definition. Let $\mathfrak{X}$ be a class of finite group such that every finite group G has a largest normal subgroup $G_{\mathfrak{X}}$ belonging to $\mathfrak{X}$. The *Baer–Suzuki width* $\text{BS}(\mathfrak{X})$ is the smallest nonnegative integer m such that

$$G_{\mathfrak{X}} = \{x \in G \mid \langle x_1, \dots, x_m \rangle \in \mathfrak{X} \text{ for all } x_1, \dots, x_m \text{ conjugate to } x\}$$

for every finite group G , or, if no such m exists, we set $\text{BS}(\mathfrak{X}) = \infty$.

The Baer–Suzuki width of the class of p -groups is 2 and that of solvable groups is 4. Gordeev, Grunewald, Kunyavskii, and Plotkin [4, Problem 1.16] proposed to study the following problem.

Problem. Study other classes $\mathfrak{X}$ of finite groups for which $\text{BS}(\mathfrak{X}) < \infty$.

Recently [5], using the classification of finite simple groups, it was proved that the Baer–Suzuki width of every complete class is finite. Here we will give an estimate of the Baer–Suzuki width for complete classes and show that it is nearly unimprovable.

1. Unimprovable estimate for the Baer–Suzuki width of a complete class

It is clear that $\text{BS}(\mathfrak{X}) = 0$ if $\mathfrak{X}$ is the class of all finite groups. Hereinafter, we assume that $\mathfrak{X}$ is a complete class distinct from the class of all finite groups. By the Cauchy embedding, there exists a greatest natural number $\Upsilon = \Upsilon(\mathfrak{X})$ such that $\text{Sym}_{\Upsilon} \in \mathfrak{X}$. We call this number Υ *the symmetric boundary* of $\mathfrak{X}$.

By slightly modifying the arguments in [5], we can prove the following assertion.

Theorem. *Let $\mathfrak{X}$ be a complete class distinct from the class of all finite groups. Denote by $\Upsilon = \Upsilon(\mathfrak{X})$ the largest integer n such that $\text{Sym}_n \in \mathfrak{X}$ and set*

$$m = \max\{11, 2(\Upsilon - 1)\}.$$

Then, for every finite group G ,

$$G_{\mathfrak{X}} = \{x \in G \mid \langle x_1, \dots, x_m \rangle \in \mathfrak{X} \text{ for all } x_1, \dots, x_m \text{ conjugate to } x\}.$$

Corollary. *Let $\mathfrak{X}$ be a complete class distinct from the class of all finite groups. Then $\text{BS}(\mathfrak{X}) \leq \max\{11, 2(\Upsilon - 1)\}$.*

We will show that the estimate $\max\{11, 2(\Upsilon - 1)\}$ is unimprovable for $\Upsilon \geq 7$.

Conclusion

An unimprovable estimate is also known for $\Upsilon = 1$: it follows from Tyutyanov’s result [6] that if $\Upsilon = 1$, then $\text{BS}(\mathfrak{X}) = 2$. It is also easy to show that the symmetric boundary of a suitable complete class can take on any natural number except 3 (the latter follows from the fact that if $\text{Sym}_3 \in \mathfrak{X}$, then $\text{Sym}_4 \in \mathfrak{X}$). Thus, the unimprovable estimate for the Baer–Suzuki width in terms of the symmetric boundary Υ of a complete class is known, except for the four cases where $\Upsilon \in \{2, 4, 5, 6\}$.

Acknowledgement. This work was supported by the Russian Science Foundation, Project No. 24-11-00127, <https://rscf.ru/en/project/24-11-00127/>.

References

- [1] H. Wielandt, Zusammengesetzte Gruppen: Hölders Programm heute, in *The Santa Cruz conf. on finite groups, Santa Cruz, 1979* (Proc. Sympos. Pure Math., **37**, Providence RI: Amer. Math. Soc., 1980).
- [2] P. Flavell, S. Guest, R. Guralnick, Characterizations of the solvable radical, *Proc. Amer. Math. Soc.*, **138**:4 (2010), 1161–1170.
- [3] N. Gordeev, F. Grunewald, B. Kunyavskii, E. Plotkin, From Thompson to Baer–Suzuki: a sharp characterization of the solvable radical, *J. Algebra*, **323**:10 (2010), 2888–2904.
- [4] N. Gordeev, F. Grunewald, B. Kunyavskii, E. Plotkin, A description of Baer–Suzuki type of the solvable radical of a finite group, *J. Pure Appl. Algebra* **213**:2 (2009) 250–258.
- [5] D. O. Revin, The Baer–Suzuki width of a complete class of finite group is finite, *Algebra i Analiz*, **37**:1 (2025), 141–176.
- [6] V. N. Tyutyanov, Kriterii neprostoty dlya konechnoi gruppy, *Izv. Gomel. gos. un-ta im. F. Skoriny (Voprosy algebry)*, **16**:3 (2000), 125–137.

Danila Revin
 Novosibirsk State University
 Novosibirsk, Russia
 e-mail: d.revin@ng.su.ru