

ON THE BAER-SUZUKI WIDTH

Danila Revin

Novosibirsk State University

Vavilov Memorial 2026, EMI,
September 18

С использованием классификации мы можем отвечать на многие вопросы не только собственно про сами конечные простые группы, но и вообще про все конечные группы. Именно этим занимаются последние годы многие лучшие специалисты по конечным группам: Майкл Ашбахер, Гари Зайтц, Мартин Либек, Роберт Гуральник, Ян Саксл, и многие другие. Я уверен, что все импликации классификации конечных простых групп, и, в еще большей степени классификации их представлений и классификации их максимальных подгрупп, как собственно в теории конечных групп, так и, тем более, в теории представлений и других разделах математики только начинают осознаваться.

With the use of the classification we can answer many questions not only about finite simple groups themselves, but in general about all finite groups. This is precisely what many of the best specialists in finite group theory have been doing in recent years: Michael Aschbacher, Gary Seitz, Martin Liebeck, Robert Guralnick, Jan Saxl, and many others. I am sure that all the implications of the classification of finite simple groups, and, to an even greater extent, of the classification of their representations and the classification of their maximal subgroups, both in finite group theory proper and, all the more, in representation theory and other branches of mathematics, are only beginning to be realized.

Nikolai A. Vavilov

 N. Gordeev, F. Grunewald, B. Kunyavskii, E. Plotkin, *A description of Baer–Suzuki type of the solvable radical of a finite group*, J. Pure Appl. Algebra, **213**, no. 2, (2009) 250–258.

We will consider **only finite groups**. The term “group” always means “finite group”.

For a group G and a prime p , let us consider

- ▶ the *solvable radical* $S(G)$, the largest normal solvable subgroup;
- ▶ the *nilpotent radical* $F(G)$ (the Fitting subgroup), the largest normal nilpotent subgroup; and
- ▶ the *p -radical* $O_p(G)$ the largest normal p -subgroup.

Theorem (Baer–Suzuki)

In a group G ,

$$F(G) = \{x \in G \mid \langle x_1, x_2 \rangle \text{ is nilpotent for all } x_1, x_2 \text{ conjugate to } x\},$$

$$O_p(G) = \{x \in G \mid \langle x_1, x_2 \rangle \text{ is a } p\text{-group for all } x_1, x_2 \text{ conjugate to } x\}$$

for every prime p .

Theorem (Gordeev, Grunewald, Kunyavskii, and Plotkin and independently Flavell, Guest, and Guralnick)

In a group G ,

$$S(G) = \{x \in G \mid \langle x_1, x_2, x_3, x_4 \rangle \text{ is solvable for all } x_1, x_2, x_3, x_4 \text{ conjugate to } x\}.$$

Definition (Gordeev, Grunewald, Kunyavskii, and Plotkin)

Let $\mathcal{X}$ be a nonempty *radical class* of groups, i.e. a class with *the Fitting property*:

$$H, K \trianglelefteq G \text{ and } H, K \in \mathcal{X} \Rightarrow HK \in \mathcal{X}.$$

Then, in every group G , there is the *$\mathcal{X}$ -radical*—the largest normal $\mathcal{X}$ -subgroup

$$G_{\mathcal{X}} := \langle H \mid H \trianglelefteq G \text{ and } H \in \mathcal{X} \rangle \in \mathcal{X}.$$

The Baer–Suzuki width $\text{BS}(\mathcal{X})$ of $\mathcal{X}$ is the smallest $m \in \mathbb{N} \cup \{0\}$ such that

$$G_{\mathcal{X}} = \{x \in G \mid \langle x_1, \dots, x_m \rangle \in \mathcal{X} \text{ for all } x_1, \dots, x_m \text{ conjugate to } x\}$$

for every group G . If such an $m \in \mathbb{N} \cup \{0\}$ does not exist then $\text{BS}(\mathcal{X}) = \infty$.

Examples

- ▶ If $\mathcal{X}$ is the class of all groups, then $\text{BS}(\mathcal{X}) = 0$;
- ▶ If $\mathcal{X}$ is the class of groups of order 1, then $\text{BS}(\mathcal{X}) = 1$;
- ▶ If $\mathcal{X}$ is the class of nilpotent groups or the class of p -groups for a prime p , then $\text{BS}(\mathcal{X}) = 2$;
- ▶ If $\mathcal{X}$ is the class of solvable groups, then $\text{BS}(\mathcal{X}) = 4$.

Problem 1 (Gordeev, Grunewald, Kunyavskii, and Plotkin)

Study other classes $\mathcal{X}$ for which $\text{BS}(\mathcal{X}) < \infty$.

Main result of the talk

If the properties of the class $\mathcal{X}$ are similar to those of the class of solvable groups, then $\text{BS}(\mathcal{X}) < \infty$.

Definition (Bender)

- ▶ A finite group G is called *quasisimple* if $G = G'$ and $G/Z(G)$ is a nonabelian simple group.
- ▶ A subnormal quasisimple subgroup of a finite group G is called a *component* of G ; the set of components of G is denoted by $\text{Comp}(G)$.
If $K, L \in \text{Comp}(G)$ and $K \neq L$, then $[K, L] = 1$.
- ▶ The *layer* of a group G is $E(G) = \langle L \mid L \in \text{Comp}(G) \rangle$.
- ▶ A finite group G is called *semisimple* if $G = E(G)$.
- ▶ The *generalized Fitting subgroup* of a group G is $F^*(G) = E(G)F(G)$.
- ▶ A finite group G is called *quasinilpotent* if $G = F^*(G)$.

The classes of quasinilpotent and semisimple groups are *radical*, but are not closed under taking subgroups.

The class of quasinilpotent groups is closed under taking normal subgroups, while the class of semisimple groups is not.

$F^*(G)$ is the quasinilpotent radical of the group G .

$E(G)$ is the semisimple radical of the group G .

A simple nonabelian group $G = F^*(G) = E(G)$ is both semisimple and quasinilpotent.

A solvable group is quasinilpotent if and only if it is solvable, and it is semisimple if and only if it is trivial.

Definition (Gordeev, Grunewald, Kunyavskii, Plotkin)

The Baer–Suzuki width $BS(\mathcal{X})$ of $\mathcal{X}$ is the smallest $m \in \mathbb{N} \cup \{0\}$ such that

$$G_{\mathcal{X}} = \{x \in G \mid \langle x_1, \dots, x_m \rangle \in \mathcal{X} \text{ for all } x_1, \dots, x_m \text{ conjugate to } x\}$$

for every group G . If such an $m \in \mathbb{N} \cup \{0\}$ does not exist then $BS(\mathcal{X}) = \infty$.

Proposition 1

Let $\mathcal{X}$ be the class of quasinilpotent groups. Then $BS(\mathcal{X}) = \infty$.

It is easy to see that $BS(\mathcal{X}) > 1$. Let $G = \text{Sym}_3$ and let x be a transposition.

“Any $m \leq 1$ ” generate a (quasi)nilpotent group, but $x \notin \text{Alt}_3 = F^*(G) = G_{\mathcal{X}}$.

$$G_{\mathcal{X}} \not\supseteq \{x \in G \mid \langle x_1, \dots, x_m \rangle \in \mathcal{X} \text{ for all } x_1, \dots, x_m \text{ conjugate to } x\}.$$

Let $m \geq 2$. We show that $BS(\mathcal{X}) > m$.

Let G be a nonabelian simple group. Then $G = G_{\mathcal{X}}$.

By the Feit–Thompson theorem G contains an involution x .

By the Baer–Suzuki theorem there exists $y \in x^G$ such that $\langle x, y \rangle$ is nonnilpotent.

But $\langle x, y \rangle$ is solvable. Therefore, $\langle x, y \rangle \notin \mathcal{X}$.

Set $x_1 = x, \quad x_2 = \dots = x_m = y$. Then

$$\langle x_1, x_2, \dots, x_m \rangle = \langle x, y \rangle \notin \mathcal{X}.$$

$$G_{\mathcal{X}} \not\supseteq \{x \in G \mid \langle x_1, \dots, x_m \rangle \in \mathcal{X} \text{ for all } x_1, \dots, x_m \text{ conjugate to } x\}.$$

- In what follows we will consider the so-called *hereditary* classes of groups, i.e. classes closed under taking subgroups:

$$G \in \mathcal{X} \text{ and } H \leq G \Rightarrow H \in \mathcal{X}.$$

It seems that the notion of the Baer–Suzuki width is the most natural just for hereditary classes.

- **The speaker is not aware of examples of radical hereditary classes of infinite Baer–Suzuki width.**
- For a hereditary class $\mathcal{X}$, the equality

$$G_{\mathcal{X}} = \{x \in G \mid \langle x_1, \dots, x_m \rangle \in \mathcal{X} \text{ for all } x_1, \dots, x_m \text{ conjugate to } x\}$$

in the definition of the Baer–Suzuki width can be replaced by the implication

$$\langle x_1, \dots, x_m \rangle \in \mathcal{X} \text{ for all } x_1, \dots, x_m \text{ conjugate to } x \Rightarrow \langle x^G \rangle \in \mathcal{X}.$$

This implication does not require the existence of the radical $G_{\mathcal{X}}$ in an arbitrary group. Moreover, the notion of the Baer–Suzuki width turns out to be meaningful not only for radical classes.

- For example, if in an arbitrary group G , the conjugacy class x^G generates a normal abelian subgroup if and only if any two elements conjugate to x generate an abelian subgroup. Thus, it is natural to consider **the Baer–Suzuki width of the class of all abelian groups** to be equal to 2.

Definition (for hereditary classes)

Let $\mathcal{X}$ be a nonempty hereditary class. *The Baer–Suzuki width $BS(\mathcal{X})$ of $\mathcal{X}$* is the smallest $m \in \mathbb{N} \cup \{0\}$ such that

$$\langle x_1, \dots, x_m \rangle \in \mathcal{X} \text{ for all } x_1, \dots, x_m \text{ conjugate to } x \Rightarrow \langle x^G \rangle \in \mathcal{X}$$

for every group G and every $x \in G$. If such an $m \in \mathbb{N} \cup \{0\}$ does not exist then $BS(\mathcal{X}) = \infty$.

Examples

- ▶ If $\mathcal{X}$ is the class of all abelian groups, then $\text{BS}(\mathcal{X}) = 2$;
- ▶ If $\mathcal{X}$ is the class of all cyclic groups, then $\text{BS}(\mathcal{X}) = 2$;
- ▶ If $\mathcal{X}$ is a class of abelian groups closed under taking
 - homomorphic images ($G \in \mathcal{X}$ and ϕ is a homomorphism with domain $G \Rightarrow \phi(G) \in \mathcal{X}$),
 - subgroups ($G \in \mathcal{X}$ and $H \leq G \Rightarrow H \in \mathcal{X}$), and
 - direct products ($G, H \in \mathcal{X} \Rightarrow G \times H \in \mathcal{X}$),then $\text{BS}(\mathcal{X}) \leq 2$; but
- ▶ There exists a class of nilpotent groups $\mathcal{X}$ closed under taking homomorphic images, subgroups, and direct products with $\text{BS}(\mathcal{X}) = \infty$.

Lemma 1 (H. Liebeck, 1962, E. Zelmanov, 1991)

Let p be a prime number. Then

$$c_{m,p} := c(B_0(m, p^2)) \geq c(\mathbb{Z}_p \wr_{\text{reg}} \mathbb{Z}_p^{m-1}) \geq (m-1)(p-1) + 1.$$

In particular, $c_{p,m}$ grows unboundedly as m grows.

Let us enumerate the primes: $p_1 = 2, \quad p_2 = 3, \quad p_3 = 5, \quad \dots$

Let $\mathcal{X}$ consist of all finite nilpotent groups of exponent free of cubes, in which the nilpotency class of every (Sylow) p_m -subgroup does not exceed c_{m,p_m} for every $m \in \mathbb{N}$. For example, $B_0(m, p_m^2) \in \mathcal{X}$.

- ▶ $\mathcal{X}$ is closed under taking subgroups, homomorphic images, and direct products, but $\text{BS}(\mathcal{X}) = \infty$.

We show that $\text{BS}(\mathcal{X}) > m$ for every integer m .

Let n be such that the nilpotency class of the group $B_0(n, p_m^2)$ is greater than c_{m,p_m} .

Set $G = B_0(n, p_m^2) \rtimes \text{Sym}_n$, and let $x = x_1, x_2, \dots, x_n$ be free generators of the group $B_0(n, p_m^2)$, permuted by Sym_n .

Then $x_1, \dots, x_n \in x^G$, whence $\langle x^G \rangle = B_0(n, p_m^2)$ is a p_m -group of class $> c_{m,p_m}$ and $\langle x^G \rangle \notin \mathcal{X}$.

But any $\leq m$ elements of x^G generate a homomorphic image of $B_0(m, p_m^2)$

and consequently an $\mathcal{X}$ -group.

Definition (Wielandt)

A nonempty class $\mathcal{X}$ of groups is said to be *complete* if $\mathcal{X}$ is closed under taking

- ▶ **homomorphic images** ($G \in \mathcal{X}$ and ϕ is a homomorphism with domain $G \Rightarrow \phi(G) \in \mathcal{X}$),
- ▶ **subgroups** ($G \in \mathcal{X}$ and $H \leq G \Rightarrow H \in \mathcal{X}$), and
- ▶ **extensions** ($H \trianglelefteq G$ and $H, G/H \in \mathcal{X} \Rightarrow G \in \mathcal{X}$).

Every complete class is both radical and hereditary: if $H, K \in \mathcal{X}$ and $H, K \trianglelefteq G$ then $HK \in \mathcal{X}$ because $H \in \mathcal{X}$ and $HK/H \cong K/(K \cap H) \in \mathcal{X}$. In particular, $\text{BS}(\mathcal{X})$ is always defined for complete $\mathcal{X}$.

Main Theorem (R., 2025)

The Baer–Suzuki width of a complete class is always finite.

Let $\mathcal{X}$ be closed under taking subgroups and homomorphic images.

Then the closedness of $\mathcal{X}$ under taking extensions implies the Fitting property (i.e. the radicality), and the Fitting property implies closedness under taking direct products:

$$\left[\begin{array}{c} \text{the closedness under} \\ \text{extensions:} \\ H \trianglelefteq G \text{ and } H, G/H \in \mathcal{X} \Rightarrow G \in \mathcal{X} \end{array} \right] \Rightarrow \left[\begin{array}{c} \text{the Fitting property:} \\ H, K \trianglelefteq G \text{ and } H, K \in \mathcal{X} \Rightarrow HK \in \mathcal{X} \end{array} \right] \Rightarrow \left[\begin{array}{c} \text{the closedness under} \\ \text{direct products:} \\ H, K \in \mathcal{X} \Rightarrow H \times K \in \mathcal{X} \end{array} \right]$$

If $\mathcal{X}$ is closed under taking direct products, it may happen that $\text{BS}(\mathcal{X}) = \infty$.

If $\mathcal{X}$ is closed under taking extensions, then $\text{BS}(\mathcal{X}) < \infty$.

Open question: What can be said about $\text{BS}(\mathcal{X})$ if $\mathcal{X}$ has the Fitting property?

Problem 2 (Kourovka Notebook, 20.81)

Assume that a class $\mathcal{X}$ is closed under taking subgroups and homomorphic images and has the Fitting property. Is $\text{BS}(\mathcal{X})$ always finite?

From this point on, $\mathcal{X}$ will always denote a complete class.

One can explicitly estimate the Baer–Suzuki width of $\mathcal{X}$ in terms of the symmetric boundary of $\mathcal{X}$.

Since $\mathcal{X}$ is nonempty and closed under subgroups, there exist integers n such that $\text{Sym}_n \in \mathcal{X}$. For example, $\text{Sym}_1 = 1 \in \mathcal{X}$. Moreover, if $\text{Sym}_n \in \mathcal{X}$ then $\text{Sym}_m \in \mathcal{X}$ for all $m \leq n$.

Definition

The *symmetric boundary* of $\mathcal{X}$ (denoted by $\Upsilon = \Upsilon(\mathcal{X})$) is the largest integer n such that $\text{Sym}_n \in \mathcal{X}$. We set $\Upsilon := \infty$ if $\text{Sym}_n \in \mathcal{X}$ for all positive integers n .

$$\text{Sym}_n \in \mathcal{X} \Leftrightarrow n \leq \Upsilon.$$

$\Upsilon = 1$ if $\mathcal{X}$ is the class of groups of order 1.

$\Upsilon = 2$ if $\mathcal{X}$ is the class of groups of odd order.

$\Upsilon \neq 3$ for all $\mathcal{X}$, because $\text{Sym}_3 \in \mathcal{X} \Rightarrow \text{Sym}_4 \in \mathcal{X}$.

$\Upsilon = n \geq 4$ if $\mathcal{X}$ is the class of groups with orders of composition factors $\leq n!/2$.

Since $\mathcal{X}$ is closed under taking subgroups, the Cayley theorem implies that

$$\Upsilon(\mathcal{X}) = \infty \Leftrightarrow \mathcal{X} \text{ is the class of all groups.}$$

In particular, $\Upsilon(\mathcal{X}) = \infty \Rightarrow \text{BS}(\mathcal{X}) = 0$.

Therefore, from now on we assume everywhere that $\mathcal{X}$ is not the class of all groups and $\Upsilon = \Upsilon(\mathcal{X}) < \infty$.

Main Theorem

For a complete class $\mathcal{X}$ different from the class of all groups, we have

$$\text{BS}(\mathcal{X}) \leq \max\{11, 2(\Upsilon - 1)\}.$$

For $\Upsilon \geq 7$, the estimate $\text{BS}(\mathcal{X}) \leq \max\{11, 2(\Upsilon - 1)\} = 2(\Upsilon - 1)$ is sharp.

Theorem (R., 2025+)

Let $n \geq 7$. Then *there exists* a complete class $\mathcal{X}$ such that

$$\Upsilon(\mathcal{X}) = n \quad \text{and} \quad \text{BS}(\mathcal{X}) \geq 2(n-1).$$

Sketch of the proof. For $n \geq 7$, set $k := n - 1$.

$G := \text{Sp}_{2k}(p)$, where $p > 2(k+1)$ is a prime.

Let $\mathcal{X}$ consist of all groups whose composition factors are isomorphic to composition factors of proper subgroups of the group G :

$$\mathcal{X} := \{H \mid \forall \text{ c. f. } S \text{ of the group } H \exists A, B < G : A \trianglelefteq B, B/A \cong S\}.$$

$\mathcal{X}$ is complete, $G \notin \mathcal{X}$, but $H \in \mathcal{X}$, for all $H < G$.

We have $\text{Sym}_{k+1} < \text{GL}_k(p) < \text{Sp}_{2k}(p)$. Therefore, $\text{Sym}_{k+1} \in \mathcal{X}$ and $\Upsilon(\mathcal{X}) \geq k+1 \geq 7$.

Lemma 2

Let $k \geq 6$ and let F be a finite field of characteristic $> 2(k+1)$.

Then $\text{Sp}_{2k}(F)$ contains no sections isomorphic to Alt_d for $d > k+1$.

$$\text{For } d \geq 5 : \quad \text{Sym}_d \in \mathcal{X} \Rightarrow \text{Alt}_d \in \mathcal{X} \Rightarrow \text{Alt}_d \text{ is a section of } G \Rightarrow d \leq k+1.$$

$$\Upsilon(\mathcal{X}) \leq k+1 = n.$$

Lemma 3

Let $x \in G$ be a transvection and $m < 2k$. Then

- ▶ $x \notin G_{\mathcal{X}}$ (otherwise $G = \langle x^G \rangle \leq G_{\mathcal{X}}$, but $G \notin \mathcal{X}$).
- ▶ If $x_1, \dots, x_m \in x^G$, then $\langle x_1, \dots, x_m \rangle < G$ and $\langle x_1, \dots, x_m \rangle \in \mathcal{X}$ (Guralnick, Saxl 2003)

$$\text{BS}(\mathcal{X}) \geq 2k = 2(n-1).$$

Theorem

Let $\mathcal{X}$ be a complete class distinct from the class of all groups, $\Upsilon = \Upsilon(\mathcal{X})$ and $m = \max\{11, 2(\Upsilon - 1)\}$. Then, in a group G , we always have

$$G_{\mathcal{X}} = \{x \in G \mid \langle x_1, \dots, x_m \rangle \in \mathcal{X} \text{ for all } x_1, \dots, x_m, \text{ conjugate to } x\}.$$

Equivalently, $\text{BS}(\mathcal{X}) \leq \max\{11, 2(\Upsilon - 1)\}$.

Lemma 4

Let $\mathcal{X}$ be complete and let $m \geq 3$ be an integer. Assume that there exist a group G such that

$$G_{\mathcal{X}} \neq \{x \in G \mid \langle x_1, \dots, x_m \rangle \in \mathcal{X} \text{ for all } x_1, \dots, x_m, \text{ conjugate to } x\}. \quad (1)$$

Take G satisfying (1) *of the smallest order*. Then G contains a subgroup L and an element x such that

- (1) $L \trianglelefteq G$;
- (2) L is nonabelian simple ;
- (3) $L \notin \mathcal{X}$;
- (4) $C_G(L) = 1$; in particular, one may assume that $L \leq G \leq \text{Aut } L$
- (5) every m elements conjugate to x generate an $\mathcal{X}$ -subgroup;
- (6) x is of prime order;
- (7) $G = \langle x, L \rangle$.

Definition (Guralnick and Saxl, 2003, Gordeev *et al.*, 2009, R., 2025)

Let L be a nonabelian simple group and $x \neq 1$ an automorphism of L . Put $G = \langle L, x \rangle$.

- ▶ $\alpha(x, L)$ is the smallest m such that, in G , some m conjugates of x generate G .

Additionally, assume that $L \notin \mathcal{X}$.

- ▶ $\beta_{\mathcal{X}}(x, L)$ is the smallest m such that, in G , some m conjugates of x generate a subgroup which does not belong to $\mathcal{X}$.

$$\beta_{\mathcal{X}}(x, L) \leq \alpha(x, L).$$

We need to prove that

$$\beta_{\mathcal{X}}(x, L) \leq \max\{11, 2(\Upsilon - 1)\}$$

for every nonabelian simple $L \notin \mathcal{X}$ and every automorphism x of prime order of L .

Proposition 2 (Guralnick and Saxl, 2003)

Let L be a nonabelian simple group and x an automorphism of prime order of L . Then one of the following assertions holds.

- ▶ $\alpha(x, L) \leq 11$ (in particular, this holds for all sporadic and exceptional groups);
- ▶ $L = \text{Alt}_n$ and $\alpha(x, L) \leq n - 1$;
- ▶ $L = L_n(q)$ or $U_n(q)$ and $\alpha(x, L) \leq n$;
- ▶ $L = S_{2n}(q)$, $O_{2n+1}(q)$, $O_{2n}^{\pm}(q)$ and $\alpha(x, L) \leq n + 3$;
- ▶ $L = S_{2n}(q)$, q is odd, or $L \cong O_{2n}^{\pm}(q)$, q is even, x is a transvection and $\alpha(x, L) \leq 2n$;
- ▶ $L = S_{2n}(q)$, q is even, x is a transvection and $\alpha(x, L) \leq 2n + 1$;
- ▶ $L = O_n^{\varepsilon}(q)$, q is odd, $\varepsilon \in \{0, +, -\}$, x is a reflection and $\alpha(x, L) \leq n$.

Proposition 3

Let $L = \text{Alt}_n$ and x an automorphism of prime order of L . Then

$$\beta_{\mathcal{X}}(x, L) \leq \max\{11, 2(\Upsilon - 1)\}. \quad (2)$$

Proof. We argue by induction on n .

Guralnick and Saxl proved that $\alpha(x, L) = n - 1$ if x is a transposition, $\alpha(x, L) = 2$ if x is an n -cycle, and $\alpha(x, L) \leq n/2$ otherwise.

We may assume that $\alpha(x, L) > 11$. Hence $n > 12$ and $\text{Aut } L = \text{Sym}_n$.

If x is an n -cycle, then $\alpha(x, L) = 2$ and (2) holds.

Assume that x is a transposition. Then $x \in \text{Sym}_{n-1}$. If $\text{Sym}_{n-1} \notin \mathcal{X}$ then (2) holds by the induction hypothesis. If $\text{Sym}_{n-1} \in \mathcal{X}$ then $n - 1 \leq \Upsilon$ and

$$\beta_{\mathcal{X}}(x, L) \leq \alpha(x, L) = n - 1 \leq \Upsilon < 2(\Upsilon - 1).$$

Otherwise, $\langle x \rangle$ acts intransitively and $x \in H = \text{Sym}_k \times \text{Sym}_{n-k}$, where $n/2 \leq k \leq n - 1$.

We may assume that the projection $\bar{x}$ of x into Sym_k is nontrivial (if not, $x \in \text{Sym}_{n-1} = \text{Sym}_{n-1} \times \text{Sym}_1$).

Assume that $k > \Upsilon$. Then $\text{Alt}_k \notin \mathcal{X}$. Since $k \geq n/2 > 5$, Alt_k is simple. By the induction hypothesis

$$\beta_{\mathcal{X}}(\bar{x}, \text{Alt}_k) \leq m, \quad m = \max\{11, 2(\Upsilon - 1)\}.$$

This implies that some m conjugates of x in $H = \text{Sym}_k \times \text{Sym}_{n-k}$ generate a subgroup K such that $\bar{K} \notin \mathcal{X}$, where $\bar{} : H \rightarrow \text{Sym}_k$ is the projection map. This means that $K \notin \mathcal{X}$ and

$$\beta_{\mathcal{X}}(x, L) \leq m.$$

Assume, $k \leq \Upsilon$. We have $\alpha(x, L) \leq n/2$. Moreover, $n/2 \leq k \leq \Upsilon$. Therefore,

$$\beta_{\mathcal{X}}(x, L) \leq \alpha(x, L) \leq n/2 \leq \Upsilon < \{11, 2(\Upsilon - 1)\}.$$

The idea of proving Proposition 3 can be implemented for classical groups as well.

NOTATION AND TERMINOLOGY

Let x be an element of G . We say that x *affects* a nonabelian simple group S and write $x \rightsquigarrow S$ if G contains some subgroups H and K such that the following assertions are valid:

- ▶ $K \trianglelefteq H$ and $H/K \cong S$,
- ▶ H and K are x -invariant, i.e. $x \in N_G(H) \cap N_G(K)$, and
- ▶ acting by conjugation, x induces a **nonidentical automorphism** $\bar{x}$ of $H/K \cong S$:

$$\bar{x} : Kh \mapsto Kh^x \quad \text{for all } h \in H.$$

In this definition, the homomorphism

$$\bar{} : N_G(H) \cap N_G(K) \rightarrow \text{Aut}(H/K)$$

is also defined.

Lemma 5

Let $x \in \text{Aut } L$ for a nonabelian simple group L . Assume that $x \rightsquigarrow S$ for a nonabelian simple $S \notin \mathcal{X}$. Then

$$\beta_{\mathcal{X}}(x, L) \leq \beta_{\mathcal{X}}(\bar{x}, S).$$

Proof. By definition, $x \in N_G(H) \cap N_G(K)$, where $G = \langle L, x \rangle$, $H, K \leq G$, $K \trianglelefteq H$ and $H/K \cong S$. Put $m = \beta_{\mathcal{X}}(\bar{x}, S)$. There are $h_1, \dots, h_m \in H$ such that

$$\overline{\langle x^{h_1}, \dots, x^{h_m} \rangle} = \langle \bar{x}^{h_1}, \dots, \bar{x}^{h_m} \rangle \notin \mathcal{X}.$$

Since $\mathcal{X}$ is closed under homomorphic images, we have $\langle x^{h_1}, \dots, x^{h_m} \rangle \notin \mathcal{X}$ and $\beta_{\mathcal{X}}(x, L) \leq m$.

Sometimes, it will be convenient for us to consider symplectic groups $S_{2n}(q)$ of characteristic 2 together with orthogonal groups $O_{2n+1}(q)$. We set

$$O_{2n+1}^*(q) = \begin{cases} O_{2n+1}(q), & q \text{ is odd,} \\ S_{2n}(q), & q \text{ is even,} \end{cases}$$

$S_{2n}^*(q)$ means the group $S_{2n}(q)$ for odd q .

For a classical simple group L , we define parameters $\mu(L)$ and $\nu(L)$ as follows.

Таблица. Definition of $\mu(L)$ and $\nu(L)$ for a classical group L

L	Restrictions	$\mu(L)$	$\nu(L)$
$L_n(q)$	$n \geq 2; q > 3$ if $n = 2$	$n + 1$	n
$U_n(q)$	$n \geq 3; q > 3$ if $n = 2$	$n + 1$	n
$O_{2n+1}^*(q)$	$n \geq 2; q > 2$ is even if $n = 2$	$2n + 1$	n
$S_{2n}^*(q)$	$n \geq 2; q$ is odd	$n + 1$	n
$O_{2n}^+(q)$	$n \geq 4$	$2n - 1$	n
$O_{2n}^-(q)$	$n \geq 4$	$2n - 1$	n

Lemma 6

In a classical group L , there are subgroups H and K such that $K \trianglelefteq H$ and $H/K \cong \text{Alt}_{\mu(L)}$. In particular, $L \notin \mathcal{X}$ if $\mu(L) > \Upsilon(\mathcal{X})$.

The idea of how to prove $\beta_{\mathcal{X}}(x, L) \leq \max\{11, 2(\Upsilon - 1)\}$ for classical L is as follows. We divide the general situation into a finite number of cases for (x, L) . In each of them, either $\alpha(x, L) \leq 11$, or $x \rightsquigarrow S$ for a classical simple group S , which is **sufficiently large with respect to L** in the sense that **the dimension of the vector space associated with S is at least half the dimension of the space associated with L** .

If $\mu(S) > \Upsilon$, then $S \notin \mathcal{X}$. In this case, we use the induction assumption and Lemma 2. If $\mu(S) \leq \Upsilon$, then we can bound $\alpha(x, L)$ by a linear function $a\Upsilon + b$. In the worst case, we obtain the function $2(\Upsilon - 1)$.

AUTOMORPHISMS OF PRIME ORDER OF CLASSICAL GROUPS

We distinguish nine types of automorphisms of prime order of classical simple groups.

- x_{uni} : an automorphism induced by unipotent element, for symplectic groups — not by transvection; we add to this case the automorphisms of $L = O_{2n}^{\pm}(q)$, q is even, from the coset Lt where t is a transvection, which are not conjugate to t ;
- x_{tra} : an automorphism induced by transvection on a group $O_{2n}^{\pm}(q)$ where q is even, (the transvection lies outside $L_n(q)$, $U_n(q)$ or $S_{2n}(q)$);
- x_{irr} : an automorphism induced by the projective image of an irreducible similarity;
- x_{red} : an automorphism induced by an inner-diagonal semisimple element leaving invariant a proper nondegenerate (except in the case of $L_n(q)$) subspace U of dimension $\geq \frac{1}{2} \dim V$ of the associated vector space V and acting non-scalarly on U . In the cases of $O_{2n+1}(q)$ and $O_{2n}^{\pm}(q)$, where q is odd, we assume that this element is not a reflection;
- x_{ref} : an automorphism of $O_{2n+1}(q)$ and $O_{2n}^{\pm}(q)$, where q is odd, induced by a reflection;
- x_{iso} : an automorphism of groups $U_n(q)$, n even, $S_{2n}(q)$ and $O_{2n}^{+}(q)$ induced by a semisimple element leaving invariant two complementary maximal totally isotropic subspaces and acting irreducibly on both of them;
- x_{fie} : a field automorphism of L modulo $\text{Inndiag } L$;
- x_{gf} : a graph-field automorphism of L modulo $\text{Inndiag } L$ (in $L_n(q)$, $O_{2n}^{\pm}(q)$);
- x_{gra} : a graph automorphism of L modulo $\text{Inndiag } L$ (in $L_n(q)$, $O_{2n}^{\pm}(q)$).

Lemma 7

Let L be a classical simple groups and x an automorphism of prime order of L . Then x is of one of types x_{uni} ,

x_{tra} , x_{irr} , x_{red} , x_{ref} , x_{iso} , x_{fie} , x_{gf} , or x_{gra} .

EXAMPLE OF A TYPICAL REASONING

- Let $L = {}^2A_{n-1}(q) = U_n(q)$ and let x be of type x_{uni} . Then either $x \rightsquigarrow U_{n-2}(q)$ or $x \rightsquigarrow L_{[n/2]}(q^2)$.

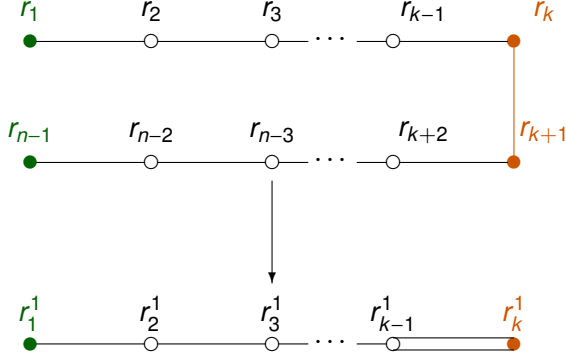


Рис.. ${}^2A_{n-1}(q)$, where
 $n-1 = 2k, k = [n/2]$.

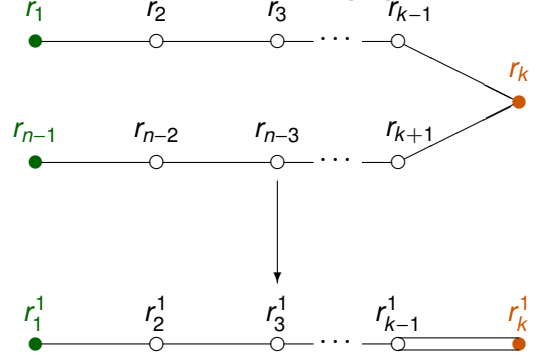


Рис.. ${}^2A_{n-1}(q)$, where
 $n-1 = 2k-1, k = [n/2]$.

Lemma 8 (Guralnick and Saxl)

Let L be a simple group of Lie type and let $x \in L$. If x is unipotent, let P_1 and P_2 be distinct parabolic maximal subgroups containing a common Borel subgroup of L with unipotent radicals U_1 and U_2 . Then x is conjugate to an element of $P_i \setminus U_i$ for $i = 1$ or $i = 2$.

Let $k = [n/2]$. Assume, P_1 and P_2 are parabolic subgroups obtained by removing one of the nodes r_1^1 and r_k^1 from the Dynkin diagram, respectively.

If x is conjugate to an element of $P_1 \setminus U_1$, then $x \rightsquigarrow {}^2A_{n-3}(q) \cong U_{n-2}(q)$.

If x is conjugate to an element of $P_2 \setminus U_2$, then $x \rightsquigarrow A_{k-1}(q^2) \cong L_{[n/2]}(q^2)$.

$L = {}^2A_{n-1}(q) = U_n(q)$ and x of type x_{uni} .

$x \rightsquigarrow S$, where $S = U_{n-2}(q)$ or $S = L_{[n/2]}(q^2)$.

Assume, $\mu(S) > \Upsilon$. Then $\text{Alt}_{\mu(S)} \notin \mathcal{X}$ and $S \notin \mathcal{X}$. By Lemma 2 and by induction assumption,

$$\beta_{\mathcal{X}}(x, L) \leq \beta_{\mathcal{X}}(\bar{x}, S) \leq \max\{11, 2(\Upsilon - 1)\}.$$

Assume, $\mu(S) \leq \Upsilon$.

- If $S = U_{n-2}(q)$, then $\mu(S) = n - 1$.
 $\mu(S) \leq \Upsilon \Rightarrow n \leq \Upsilon + 1$,
 $\beta_{\mathcal{X}}(x, L) \leq \alpha(x, L) \leq n \leq \Upsilon + 1 \leq \max\{11, 2\Upsilon + 1\}$.
- If $S = L_{[n/2]}(q^2)$, then $\mu(S) = [n/2] + 1$.
 $\mu(S) \leq \Upsilon \Rightarrow n/2 + 1 \leq \Upsilon \Leftrightarrow n \leq 2(\Upsilon - 1)$,
 $\beta_{\mathcal{X}}(x, L) \leq \alpha(x, L) \leq n \leq 2(\Upsilon - 1) \leq \max\{11, 2\Upsilon + 1\}$.

L	type of x	$\alpha(x, L) \leq$	$S, x \rightsquigarrow S$	$\mu(S)$	$\mu(S) \leq \Upsilon \Rightarrow$ $v(L) = n \leq$	$\mu(S) \leq \Upsilon \Rightarrow$ $\alpha(x, L) \leq$
$U_n(q)$	x_{uni}	n	$U_{n-2}(q)$	$n - 1$	$\Upsilon + 1$	$\Upsilon + 1$
			$L_{[n/2]}(q^2)$	$[n/2] + 1$	$2(\Upsilon - 1)$	$2(\Upsilon - 1)$

Lemma 9

Let L be a classical simple groups and x an automorphism of prime order of L . Then the following statements hold:

- ▶ If x is of type x_{irr} , then $\alpha(x, L) \leq 3$;
- ▶ If x is of type different from x_{irr} then $x \leadsto S$, where S is one of classical simple groups in the fourth column of Tables below. Moreover, $\mu(S)$ is at least the number in the fifth column. If $\mu(S) \leq \Upsilon$ then $\nu(L)$ and $\alpha(x, L)$ do not exceed the expression in the sixth and seventh columns, respectively. Except for the case when $L = S_{2n}(q)$ and x is of type x_{red} , the value in the seventh column does not exceed $2(\Upsilon - 1)$.

Таблица. Types x_{uni} and x_{tra}

L	type of x	$\alpha(x, L) \leq$	$S, x \leadsto S$	$\mu(S) \geq$	$\mu(S) \leq \Upsilon \Rightarrow \nu(L) = n \leq$	$\mu(S) \leq \Upsilon \Rightarrow \alpha(x, L) \leq$
$L_n(q)$	x_{uni}	n	$L_{n-1}(q)$	n	Υ	Υ
$U_n(q)$	x_{uni}	n	$U_{n-2}(q)$	$n - 1$	$\Upsilon + 1$	$\Upsilon + 1$
			$L_{[n/2]}(q^2)$	$[n/2] + 1$	$2(\Upsilon - 1)$	$2(\Upsilon - 1)$
$O_{2n+1}^*(q)$	x_{uni}	$n + 3$	$O_{2n-1}^*(q)$	$2n - 1$	$(\Upsilon + 1)/2$	$(\Upsilon + 7)/2$
			$L_n(q)$	$n + 1$	$\Upsilon - 1$	$\Upsilon + 2$
	x_{tra}	$2n + 1$	$O_{2n}^\pm(q), q$ is even	$2n - 1$	$(\Upsilon + 1)/2$	$\Upsilon + 2$
$S_{2n}^*(q)$	x_{uni}	$n + 3$	$S_{2(n-1)}^*(q)$	n	Υ	$\Upsilon + 3$
			$L_n(q)$	$n + 1$	$\Upsilon - 1$	$\Upsilon + 2$
	x_{tra}	$2n$	$S_{2(n-1)}^*(q)$	n	Υ	2Υ
$O_{2n}^\varepsilon(q)$	x_{uni}	$n + 3$	$O_{2(n-1)}^\pm(q)$	$2n - 3$	$(\Upsilon + 3)/2$	$(\Upsilon + 9)/2$
			$L_n(q),$ if $\varepsilon = +$	$n + 1$	$\Upsilon - 1$	$2(\Upsilon - 1)$
			$L_{n-1}(q),$ if $\varepsilon = -$	n	Υ	2Υ
	x_{tra}	$2n$	$O_{2n-1}^*(q), q$ is even	$2n - 1$	$(\Upsilon + 1)/2$	$\Upsilon + 1$

Таблица. Types x_{red} , x_{iso} , and x_{ref}

L	type of x	$\alpha(x, L) \leq$	$S, x \rightsquigarrow S$	$\mu(S) \geq$	$\mu(S) \leq \Upsilon \Rightarrow \nu(L) = n \leq$	$\mu(S) \leq \Upsilon \Rightarrow \alpha(x, L) \leq$
$L_n(q)$	x_{red}	n	$L_k(q), \quad n/2 \leq k \leq n-1$	$n/2 + 1$	$2(\Upsilon - 1)$	$2(\Upsilon - 1)$
$U_n(q)$	x_{red}	n	$U_k(q), \quad n/2 \leq k \leq n-1$	$n/2 + 1$	$2(\Upsilon - 1)$	$2(\Upsilon - 1)$
	x_{iso}	n	$L_{n/2}(q^2), \quad n \text{ is even}$	$n/2 + 1$	$2(\Upsilon - 1)$	$2(\Upsilon - 1)$
$O_{2n+1}(q)$	x_{red}	$n + 3$	$O_{2k+1}(q), \quad n/2 \leq k \leq n-1$	$n + 1$	$\Upsilon - 1$	$\Upsilon + 2$
			$O_{2k}^\pm(q), \quad n/2 \leq k \leq n$	$n - 1$	$\Upsilon + 1$	$\Upsilon + 4$
	x_{ref}	$2n + 1$	$O_{2n}^\pm(q), \quad q \text{ is odd}$	$2n - 1$	$(\Upsilon + 1)/2$	$\Upsilon + 2$
$S_{2n}(q)$	x_{red}	$n + 3$	$S_{2k}(q), \quad n/2 \leq k \leq n-1$	$n/2 + 1$	$2(\Upsilon - 1)$	$2\Upsilon + 1$
	x_{iso}	$n + 3$	$L_n(q)$	$n + 1$	$\Upsilon - 1$	$\Upsilon + 2$
$O_{2n}^\varepsilon(q)$	x_{red}	$n + 3$	$O_{2k}^\pm(q), \quad n/2 \leq k \leq n-1$	$n - 1$	$\Upsilon + 1$	$\Upsilon + 4$
			$O_{2k+1}(q), \quad n/2 + 1 \leq k \leq n-1$	$n + 3$	$\Upsilon - 3$	Υ
	x_{iso}	$n + 3$	$L_n(q), \quad \varepsilon = +$	$n + 1$	$\Upsilon - 1$	$\Upsilon + 2$
	x_{ref}	$2n$	$O_{2n-1}(q), \quad q \text{ is odd}$	$2n - 1$	$(\Upsilon + 1)/2$	$\Upsilon + 1$

Таблица. Types x_{fie} , x_{gf} , and x_{gra}

L	type of x	$\alpha(x, L) \leq$	$S, x \rightsquigarrow S$	$\mu(S) \geq$	$\mu(S) \leq \Upsilon \Rightarrow$ $\nu(L) = n \leq$	$\mu(S) \leq \Upsilon \Rightarrow$ $\alpha(x, L) \leq$
$L_n(q)$	x_{fie}	n	$L_{n-1}(q)$	n	Υ	Υ
	x_{gf}	n	$U_n(q_0), \quad q = q_0^2$	$n+1$	$\Upsilon-1$	$\Upsilon-1$
	x_{gra}	n	$L_{n-1}(q)$	n	Υ	Υ
			$L_{n-2}(q)$	$n-1$	$\Upsilon+1$	$\Upsilon+1$
$U_n(q)$	x_{fie}	n	$U_{n-1}(q)$	n	Υ	Υ
	x_{gra}	n	$U_{n-1}(q)$	n	Υ	Υ
			$U_{n-2}(q)$	$n-1$	$\Upsilon+1$	$\Upsilon+1$
$S_{2n}(q)$	x_{fie}	$n+3$	$S_{2(n-1)}(q)$	n	Υ	$\Upsilon+3$
$O_{2n+1}(q)$	x_{fie}	$n+3$	$O_{2n-1}(q)$	$2n-1$	$(\Upsilon+1)/2$	$(\Upsilon+7)/2$
$O_{2n}^+(q)$	x_{fie}	$n+3$	$O_{2n-1}(q)$	$2n-1$	$(\Upsilon+1)/2$	$(\Upsilon+7)/2$
	x_{gf}	$n+3$	$O_{2n}^-(q_0), \quad q = q_0^2$	$2n-1$	$(\Upsilon+1)/2$	$(\Upsilon+7)/2$
	x_{gra}	$n+3$	$O_{2n-1}(q)$	$2n-1$	$(\Upsilon+1)/2$	$(\Upsilon+7)/2$
			$O_n(q^2), \quad n \text{ is odd}$	n	Υ	$\Upsilon+3$
$O_{2n}^-(q)$	x_{fie}	$n+3$	$O_{2n-1}(q)$	$2n-1$	$(\Upsilon+1)/2$	$(\Upsilon+7)/2$
	x_{gra}	$n+3$	$O_{2n-1}(q)$	$2n-1$	$(\Upsilon+1)/2$	$(\Upsilon+7)/2$
			$O_n(q^2), \quad n \text{ is odd}$	n	Υ	$\Upsilon+3$

Suppose $L = S_{2n}(q)$ and x is of type x_{red} .

Then $x \rightsquigarrow S$, where $S_{2k}(q)$ and $k \geq n/2$.

Guralnick and Saxl showed that, in this case,

$$\alpha(x, L) \leq \begin{cases} k + 4 & \text{if } k \neq n/2, \\ k + 5 & \text{if } k = n/2. \end{cases}$$

Thus, if $k + 1 = \mu(S) \leq \Upsilon$, then

$$\beta_{\mathcal{X}}(x, L) \leq \alpha(x, L) \leq k + 5 \leq \Upsilon + 4 \leq 2(\Upsilon - 1).$$

The work is supported by the Russian Science Foundation, Project No. 24-11-00127
<https://rscf.ru/en/project/24-11-00127/>.

REFERENCES

-  R. Baer, *Engelsche Elemente Noetherscher Gruppen*, Math. Ann., (1957), **133**, 256–270.
-  M. Suzuki, *Finite groups in which the centralizer of any element of order 2 is 2-closed*, Ann. Math. (2), (1965) **82**, 191–212.
-  P. Flavell, S. Guest, R. Guralnick, *Characterizations of the solvable radical*, Proc. Amer. Math. Soc., (2010) **138**, no. 4, 1161–1170.
-  N. Gordeev, F. Grunewald, B. Kunyavskiĭ, E. Plotkin, *A description of Baer–Suzuki type of the solvable radical of a finite group*, J. Pure Appl. Algebra, (2009) **213**, no. 2, 250–258.
-  N. Gordeev, F. Grunewald, B. Kunyavskiĭ, E. Plotkin, *From Thompson to Baer–Suzuki: a sharp characterization of the solvable radical*, J. Algebra, (2010), **323**, no. 10, 2888–2904.
-  H. Wielandt, *Zusammengesetzte Gruppen endlicher Ordnung. Lecture notes, Math. Inst. Univ. Tübingen (1963/64)*, 607–655, in book: Helmut Wielandt. Mathematische Werke. V. 1, de Gruyter, Berlin. 1994.
-  H. Liebeck, *Concerning nilpotent wreath products*, Math. Proc. Cambridge Philos. Soc., **58**, no 3 (1962), 443–451.
-  E. Zelmanov, *A Solution of the Restricted Burnside Problem for 2-group*, Mathematics of the USSR-Sbornik, **72**, no. 2 (1992), 543–565.
-  R. Guralnick, J. Saxl, *Generation of finite almost simple groups by conjugates* J. Algebra, (2003), **268**, 519–571.
-  D. O. Revin, *The Baer–Suzuki width of a complete class of finite group is finite*, Algebra i Analiz, **37**, no. 1 (2025) 141–176

THANK YOU FOR YOUR ATTENTION.