

Waring-like Problems on Algebras

Vavilov Memorial 2026 (September 17-19, 2026)

**Euler International Mathematical Institute,
St. Petersburg, RUSSIA**

Anupam Singh (IISER Pune, India)

email : anupam@iiserpune.ac.in

Outline:

- ▶ Word maps on groups: Motivation.
- ▶ Polynomial maps on (non)-associative algebras.
- ▶ Polynomial maps with constants on algebras.
- ▶ Lie maps on Lie algebras.

Waring Problem

The Waring (1770) problem in number theory:

Given $k > 1$, does there exist $g(k)$ (smallest) such that every positive integer is a sum of $g(k)$ many k -th powers of non-negative integers.

That is, find $g(k)$ (smallest) such that

$$f(X_1, \dots, X_{g(k)}) = X_1^k + \dots + X_{g(k)}^k$$

is surjective on $\mathbb{N}$.

The Waring Problem in Number Theory

For example, the four-square theorem says $g(2) = 4$, i.e., every positive integer is a sum of four squares.

$$g(3) = 9, g(4) = 19, g(5) = 37, g(6) = 73 \text{ etc.}$$

For certain k one has $g(k) = 2^k + \lfloor (3/2)^k \rfloor - 2$.

For example, see the survey articles

- ♣ "Waring's problem: a survey" by Vaughan and Wooley.
- ♣ "Waring problem as an issue of polynomial computer algebra" by Nikolai Vavilov.

Generalisation of Waring Problem

- ▶ Waring-like problem for groups.
- ▶ Matrix Waring problem.
- ▶ Waring problem over certain fields and rings.
- ▶ Over Lie algebra, central simple algebras, and some non-associative algebras.

Waring-like Problem on Groups

Word Maps on Groups

- ▶ G a (finite) group, w an element of the free group $\mathcal{F}_d$ in d variables $X_1, \dots, X_d$.
- ▶ **Word map**
 $w: G^d \rightarrow G$ **given by** $(g_1, \dots, g_d) \mapsto w(g_1, \dots, g_d)$.
- ▶ How big is the image $w(G) := w(G^d) \subset G$?
- ▶ In particular, is w surjective?
- ▶ What's the width of $\langle w(G) \rangle$ with respect to $w(G)$?
- ▶ When G is finite, estimate $\frac{|w(G)|}{|G|}$?

Some Classic Examples

1. **Commutator map:** When $w(X_1, X_2) = X_1 X_2 X_1^{-1} X_2^{-1}$ the map $w: G \times G \rightarrow G$ given by

$$(g, h) \mapsto ghg^{-1}h^{-1}$$

is called the commutator map.

2. **Power map:** When $w(X) = X^M$ the map $w: G \rightarrow G$ given by

$$g \mapsto g^M$$

is called a power map.

Ore's Conjecture

Ore (1951) conjectured that the commutator map is surjective for all non-Abelian finite simple groups.

Now this conjecture is proved in the affirmative.

- ▶ Ore himself proved it for the alternating groups in 1951.
- ▶ Thompson for SL_n in 1961.
- ▶ Ellers and Gordeev proved for all finite simple groups of Lie type (except for some of small size) in 1998.
- ▶ Liebeck, O'Brien, Shalev, and Tiep completed the proof in 2010.
- ▶ Liao, Wang, and Zhang (2026) have announced recently that Thompson's conjecture is solved.

Waring-like Problem in Groups

Waring-like problem for non-Abelian finite simple (and quasi-simple) groups:

Given M , does there exist $f(M)$ such that, the word map

$$X_1^M \cdots X_{f(M)}^M$$

is surjective on all non-Abelian finite simple (quasi-simple) groups?

Borel Dominance Theorem (Borel 1983)

G a simple algebraic group and $w \in F_d$, a non-trivial word.

Then, the word map defined by w is a dominant morphism.

Thus, for simple algebraic groups, $f(M) = 2$. That is, the map $X^M Y^M$ will be surjective.

Summary - Waring-like Problem

Mainly from the work of Larsen, Shalev, and Tiep, it follows that for **large enough** G , we have

- ▶ $w(G)^2 = G$ when G is finite simple and
- ▶ $w(G)^3 = G$ when G is finite quasi-simple.

♣ See the survey article by Shalev titled “Some results and problems in the theory of word maps”.

Surjectivity of certain maps on FSG

In a series of four papers by Liebeck, O'Brien, Shalev, and Tiep, and another paper together with Guralnick, the following results are proved:

- ▶ Every element of every FSG is a product of two squares.
- ▶ Every element of every FSG is a product of two n -th powers where $n = p^k$ or $p^a q^b$. It is not true when n is a product of three prime powers.

Polynomial Maps over Rings and Algebras

Polynomial Maps over Rings

- ▶ Let $\mathfrak{R}$ be an associative ring (or algebra) with 1. Does the Waring problem have a solution over the ring $\mathfrak{R}$?
- ▶ Consider $f(X_1, \dots, X_n) \in \mathfrak{R}[X_1, \dots, X_n]$ and define

$$\omega: \mathfrak{R}^n \rightarrow \mathfrak{R}$$

given by

$$(a_1, \dots, a_n) \mapsto f(a_1, \dots, a_n).$$

- ▶ Is $\mathfrak{R} = \underbrace{\omega(\mathfrak{R}) + \dots + \omega(\mathfrak{R})}_{G(\omega)}$? Find smallest $G(\omega)$!!

- ▶ $\mathfrak{R}$ need not be commutative and the polynomial f involves non-commuting variables.

Waring Problem over Rings

- ▶ For example, when $f(X) = X^k$ the question would be if every element of $\mathfrak{R}$ is a sum of finitely many k -th powers?
- ▶ If answer is yes, what is the smallest $G(k)$ which does this job.
- ▶ Some results are known when $\mathfrak{R} = \mathbb{Z}, \mathbb{F}_q, \mathbb{M}_n(\mathbb{K})$ etc.

Waring Problem over Rings

Some nice results are known when

- ▶ $\mathfrak{R}$ is a function field (for example, by Car),
- ▶ polynomial rings (see “On the Waring problem for polynomial rings” by Froberg, Ottaviani and Shapiro),
- ▶ Algebras over fields (“Waring’s problem for algebras over fields” by Vaserstein), etc.

Matrix Waring Problem

Matrix Waring Problem

- ▶ Let R be a commutative ring with 1, and $k, n \geq 2$.
- ▶ The matrix Waring problem asks if we can write every element of $M_n(R)$ as a sum of “finitely” many k -th powers.
- ▶ If yes, what is the smallest number required? If not, which matrices can be expressed in this way?
- ▶ This involves the work of Newman, Vaserstein, Richman, Katre, A. Garge, Bresar, and Semrl, etc.
- ▶ See the survey article by Barai titled "The Waring Problem for Matrices - Past, Present and Future".

Waring Problem for Matrices - some known results

- ▶ (Newman 1985) Every element of $M_n(\mathbb{Z})$, $n > 1$ is a sum of at most $8 - (-1)^n$ squares.
- ▶ (Vaserstein 1985) For any R , commutative ring with 1, every element is a sum of $P_n(R)$ squares where $P_3(R) = 3$, $P_4(R) = 4$ and $P_n(R) = \frac{1}{2}(11 - (-1)^n)$ for $n \geq 5$.
- ▶ (Richman 1985) Let F be a field of characteristic $\neq 2$. Then, $A \in M_n(F)$ is a sum of two squares if A is not a scalar matrix.

Waring Problem for Matrices - some known results

- ▶ (Richman) Let $n \geq k \geq 1$. Then, TFAE
 1. M is a sum of k th powers in $M_n(R)$.
 2. M is a sum of seven k th powers in $M_n(R)$.
 3. $M \in M_n(R)$ and for every prime power p^e dividing k , there are $\alpha_0, \dots, \alpha_e \in R$ such that
$$\text{Tr}(M) = \alpha_0^{p^e} + p\alpha_1^{p^{e-1}} + \dots + p^e\alpha_e.$$

- ▶ (Katre, A. Garge 2013) Let $n, k \geq 2$. Then TFAE
 1. M is a sum of k -th powers of matrices in $M_n(R)$.
 2. $\text{Tr}(M)$ is a sum of traces of k -th powers of matrices in $M_n(R)$.
 3. $\text{Tr}(M)$ is the subgroup of R generated by the traces of k -th powers of matrices in $M_n(R)$.
 4. $\text{Tr}(M)$ is a sum of traces of k th powers of matrices in $M_n(R) \pmod{k!R}$.

Waring Problem for Matrix Algebras - more results

- ▶ In some recent work, Bresar and Semrl (Israel Journal of Maths 22, Bull. Lond. Math Soc. 23) have proved several nice results.
- ▶ Let $A = M_n(\mathbb{C})$, with $n \geq 2$. Let $f \in \mathbb{C} \langle X \rangle$, polynomials in the free algebra on $X = \{X_1, X_2, \dots\}$, which is neither an identity nor central polynomial of A .
- ▶ Then, every trace-zero matrix in A can be written as a sum of two matrices from $f(A) - f(A)$.
- ▶ See the survey article by Bresar and Martinez titled “Waring problems across algebras” 2026 for several open questions.

Larsen's Question

- ▶ Larsen proposed that one should be able to get analogous results as in the case of finite simple groups.
- ▶ In other words, if R denotes a commutative ring with unity, then the Matrix Waring Problem would be to address whether matrices over R can be expressed as a sum of **two** k -th powers (of matrices) when R is large enough.
- ▶ The classic scenario is when R is a finite field $\mathbb{F}_q$, with q sufficiently large.

Some results

We have the following:

Theorem (Kishore and S..., Israel J. Math 2024)

*For all integers $k \geq 1$, there exists a constant C_k depending only on k , such that for all $q > C_k$ and for all $n \geq 1$ every matrix in $M_n(\mathbb{F}_q)$ is a sum of **two** k -th powers.*

Some results

In some recent work, we have proved the following:

Theorem (Choudhary, Kishnani, S..., Arxiv 2026)

Let $\mathcal{O}_\ell$ be a finite principal local ring of length ℓ . Let $\mathfrak{m}$ be the maximal ideal and the residue field $\mathcal{O}_\ell/\mathfrak{m} \cong \mathbb{F}_q$. Assume that -1 is a k -th power of an element in $\mathbb{F}_q$ and $\text{char}(\mathbb{F}_q) \nmid k$.

*Then, there exists a constant $C_{k,n}$ (depending only on n and k), such that for all $q > C_{k,n}$ **every matrix of $M_n(\mathcal{O}_\ell)$ is a sum of two k -th powers in $M_n(\mathcal{O}_\ell)$.***

Polynomial maps on algebras

Definitions

- ▶ $\mathbb{K}$ a field and $\mathbb{K}[x_1, \dots, x_n]$ polynomials in non-commuting variables.
- ▶ Given $\mathfrak{R}$ an algebra, $p \in \mathbb{K}[x_1, \dots, x_n]$, we have the map $p: \mathfrak{R}^n \rightarrow \mathfrak{R}$ given by $(a_1, \dots, a_n) \mapsto p(a_1, \dots, a_n)$.
- ▶ Such maps are called polynomial maps.
- ▶ p is a **multilinear polynomial** if each x_i appears with degree at most 1 in each term, e.g.,

$$\sum_{\sigma \in S_n} a_{\sigma} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}.$$

Properties of the image

- ▶ $Im(p)$ is closed under conjugation by invertible elements $\mathfrak{R}^\times$.
- ▶ $p(0) \in Im(p)$.
- ▶ The general question is to determine the image of p on $\mathfrak{R}$ as p varies.
- ▶ Chuang (1990) proved that any subset $S \subset M_n(\mathbb{F}_q)$ containing 0 is the image of a polynomial with constant term 0 if and only if S is closed under conjugation.

L'vov-Kaplansky conjecture

- ▶ **LK conjecture:** Let p be a multilinear polynomial. Then $Im(p)$ on $M_n(\mathbb{K})$, when $\mathbb{K}$ is infinite, is a vector space. In fact, the set of values of p is either $\{0\}$, $\mathbb{K}$, $sl_n(\mathbb{K})$ or $M_n(\mathbb{K})$.
- ▶ This is proved for $n = 2$ (2012), and 3 (2016) by Kanel-Belov, Malev and Rowen.
- ▶ Some partial results are known for higher n and particular fields.
- ▶ Images of graded polynomials on matrix algebras are studied by Centrone and de Mello (2022).

L'vov-Kaplansky conjecture

- ▶ **General Question:** Let p be a multilinear polynomial. Let $\mathcal{A}$ be an algebra (e.g., a central simple algebra) over $\mathbb{K}$. Determine the $Im(p)$ on $\mathcal{A}$?
- ▶ Malev has done this for $\mathbb{H}$.
- ▶ Work on the image of polynomials over triangular matrix algebra is done by Wang, Liu, Bai, Zhou, Luo, Fagundes, Gargate, de Mello, Panja, and Prasad.
- ▶ Malev, Yavich, Shayer (2022) have obtained results for multilinear polynomials on certain Jordan algebras over $\mathbb{R}$.

Diagonal map on matrix algebra

- ▶ For $n \geq 2$, consider the polynomial map on $M_n(K)$ for $f(X_1, \dots, X_m) = \delta_1 X^{k_1} + \delta_2 X^{k_2} + \dots + \delta_m X^{k_m}$ over K .
- ▶ Panja, Saini, S..., "Surjectivity of polynomial maps on matrices" in the European Journal of Mathematics, 2025, has the following result:
- ▶ When $K = \mathbb{R}$ and $m = 2$, it is surjective except when n is odd, $\delta_1 \delta_2 > 0$, and k_1, k_2 are both even (in that case, the image misses negative scalars), and the map is surjective for $m \geq 3$.
- ▶ For example, the image of $f = X_1^2 + X_2^4$ on $M_3(\mathbb{R})$ misses negative scalars.

Polynomial maps with constants

Polynomial maps with constants

- ▶ Given $\mathfrak{R}$ a ring, $p \in \mathfrak{R} \langle x_1, \dots, x_n \rangle$, we have the map $p: \mathfrak{R}^n \rightarrow \mathfrak{R}$ given by $(a_1, \dots, a_n) \mapsto p(a_1, \dots, a_n)$.
- ▶ Such maps are called **polynomial maps with constants** (terminology borrowed from analogous questions in group theory, e.g., work of Gordeev, Kunyavskii, and Plotkin).
- ▶ For example,

$$p = \alpha_1 X_1^{n_1} + \dots + \alpha_m X_m^{n_m},$$

$$p = \alpha_1 X_1 X_2 - \alpha_2 X_2 X_1,$$

$$p = \alpha_1 X_1^{n_1} X_2^{n_2} + X_1^{m_1} \alpha_2 X_2^{m_2} + X_1^{r_1} X_2^{r_2} \alpha_3.$$

Diagonal map on matrix algebra over $\mathbb{H}$

- ▶ For $n \geq 2$, consider the polynomial map with constants on $\mathfrak{R} = M_n(\mathbb{H})$ for $f(X_1, \dots, X_m) = \delta_1 X^{k_1} + \delta_2 X^{k_2} + \dots + \delta_m X^{k_m}$ over $\mathbb{H}$.
- ▶ Panja, Saini, S..., "Surjectivity of polynomial maps on matrices" in the European Journal of Mathematics, 2025, has the following:
- ▶ We show that on $M_n(\mathbb{H})$ the diagonal map is surjective for $m \geq 2$, where $\mathbb{H}$ is the algebra of Hamilton's quaternions.

Polynomial maps with constants: results

- ▶ Consider $\mathfrak{R} = M(2, K)$ over algebraically closed field K .
- ▶ Panja, Saini, S... explored the image of polynomials with constants ("Images of polynomial maps with constants" Mathematika 2025).
- ▶ We consider images of the polynomial maps
 - $AX^{k_1} + BY^{k_2}$ and
 - $AXY - BYX$ generalized commutator map,where A, B are nonzero elements of $M(2, K)$.
- ▶ We show that the images of these maps are vector spaces.

Polynomial maps with constants: results

Theorem (Saini, S... Linear Algebra and Appl. 2026)

Let K be an algebraically closed field of characteristic 0, and $k \geq 2$ an integer. For $A \in \mathrm{GL}_n(K)$ and $B \in M_n(K) \setminus \{0\}$, consider the polynomial map $\omega: M_n(K) \times M_n(K) \rightarrow M_n(K)$ given by

$$\omega(X, Y) = AX^k + BY^k.$$

Let r_0 be the nullity of B . Then we have the following:

1. If $r_0 \leq 1$, then ω is surjective on $M_n(K)$.
 2. If $n \leq k(r_0 - 1)$, then ω is not surjective on $M_n(K)$.
- The second statement is if and only if for $n = 3$ and 4 that helps classify the images depending on the pair A, B .

Steps used in the proof

- ▶ We compute the images, when $n = 2$, by fixing a simultaneous conjugate pair for A and B , and it turns out to be surjective in most cases.
- ▶ In general, when $B \in \mathrm{GL}_n(K)$, the map $\tilde{\omega}(X, Y) = X^k + BY^k$ is surjective on $M_n(K)$.
- ▶ For $J \in M_n(K)$ which has at most one nilpotent Jordan block, i.e., $r_0(J) \leq 1$, then the map $X^k + JY^k$ is surjective on $M_n(K)$.
- ▶ Suppose $J \in M_n(K)$ with $r_0(J) \geq 2$. If $n \leq k(r_0 - 1)$ then the map $X^k + JY^k$ is not surjective on $M_n(K)$.

Polynomial maps with constants: results

- ▶ Let $\mathcal{O}(K)$ be the split octonion algebra. For positive integers $k_1, k_2 \geq 2$, we study surjectivity of the map $AX^{k_1} + BY^{k_2}$ when $A, B \in \mathcal{O}(K)$. (Panja, Saini, S... "Polynomial maps with constants on split octonion algebras" Comm. Algebra 2025)
- ▶ For this, we use the orbit representatives of the $G_2(K)$ -action on $\mathcal{O}(K) \times \mathcal{O}(K)$ for the tuple (A, B) (already done by Lopatin and Zubkov (2025)) and characterize the ones that give a surjective map.

Matrix algebra over local rings

Polynomial maps on matrix algebra over local rings

- ▶ Let $\mathcal{O}_l$ be a finite local principal ideal ring of length l (e.g. $\mathbb{Z}/p^l\mathbb{Z}$ and $\mathbb{F}_p[t]/\langle t^l \rangle$) with the residue field $\mathbb{F}_q$.
- ▶ Suppose a polynomial map on $M_n(\mathbb{F}_q)$ is surjective, does this imply the map is surjective on $M_n(\mathcal{O}_l)$?
- ▶ In general, the question would be whether we can lift a solution.

$$\begin{array}{ccc} M_n(\mathcal{O}_l) \times \cdots \times M_n(\mathcal{O}_l) & \xrightarrow{f(x_1, \dots, x_m) = A ?} & M_n(\mathcal{O}_l) \\ \downarrow & & \downarrow \\ M_n(\mathbb{F}_q) \times \cdots \times M_n(\mathbb{F}_q) & \xrightarrow{f(x_1, \dots, x_m) = \bar{A}} & M_n(\mathbb{F}_q) \end{array}$$

Polynomial maps on matrix algebra over local rings

- ▶ We (Panja, Roy, S...) have examined the power map and have successfully solved this for $l = 2$ for most matrices.
- ▶ The idea is to get Hensel's lifting criteria for a matrix equation $f(x_1, \dots, x_r) = A$.
- ▶ Let $\mathcal{O}_2$ be a finite local principal ideal ring of length two, with residue field k having characteristic p , an odd prime.
- ▶ Let $L > 0$ be an integer coprime to p .
- ▶ Let $A \in M_n(\mathcal{O}_2)$ be regular semisimple, and let $F(t) \in \mathcal{O}_2[t]$ be monic of degree d .

Power maps on matrix algebra over local rings

- ▶ **Hensel's Lemma:** Suppose there exists $\tilde{B} \in M_n(\mathbb{F}_q)$ such that $\overline{F}(\tilde{B}) = \overline{A}$ and $\overline{F}'(\tilde{B}) \in GL_n(\mathbb{F}_q)$ where $F'(t)$ is the formal derivative of $F(t)$ in $\mathcal{O}_2[t]$. Then, there exists $B \in M_n(\mathcal{O}_2)$ with $\overline{B} = \tilde{B}$ and $F(B) = A$.
- ▶ Further, a regular semisimple element $A \in GL_n(\mathcal{O}_2)$ is an L -th power if and only if each fundamental irreducible factor of the characteristic polynomial $\chi_{\mathcal{O}_2, A}(t)$ is an L -power polynomial.
- ▶ And, a compatible cyclic element $A \in GL_n(\mathcal{O}_2)$ is an L -th power if and only if each fundamental irreducible factor of $\chi_{\mathcal{O}_2, A}(t)$ is an L -power polynomial.

Lifting commuting pair of solution

- ▶ $\hat{\mathcal{O}}$ a local principal ideal ring, complete, with residue field k of characteristic $\neq 2$.
- ▶ $A \in M_n(\hat{\mathcal{O}})$, a cyclic element, and $F \in \hat{\mathcal{O}}[x_1, x_2, \dots, x_m]$ be a monic polynomial of degree d .
- ▶ Suppose there is a commuting solution $f(\tilde{\mathbf{B}}) = \bar{A}$, where $\tilde{\mathbf{B}} = (\tilde{B}_1, \tilde{B}_2, \dots, \tilde{B}_m) \in M_n(k)^m$ satisfying $\tilde{B}_i$ commute pairwise.
- ▶ Further, assume that there exists some r , $1 \leq r \leq m$, such that $\frac{\partial f}{\partial x_i}|_{\tilde{\mathbf{B}}} \in GL_n(k)$ for $i = 1, 2, \dots, r$ and $\frac{\partial f}{\partial x_i}|_{\tilde{\mathbf{B}}} = \mathbf{0} \in M_n(k)$ for $i = r + 1, r + 2, \dots, m$.
- ▶ Then, there exists a solution $F(\mathbf{B}) = A$ where $\mathbf{B} = (B_1, B_2, \dots, B_m) \in M_n(\hat{\mathcal{O}})^m$ such that $\bar{\mathbf{B}} = \tilde{\mathbf{B}}$.

Engel maps on SL_n

- ▶ The m -th Engel word $e_m(x, y)$ in $\mathcal{F}_2$ is defined recursively as follows: $e_1(x, y) = [x, y] = xyx^{-1}y^{-1}$, $e_2(x, y) = [[x, y], y]$ and $e_m(x, y) = [e_{m-1}(x, y), y]$ for $m \geq 2$.
- ▶ The induced m -th Engel word map is $e_m^{\mathcal{O}}: \mathrm{SL}_n(\mathcal{O}) \times \mathrm{SL}_n(\mathcal{O}) \rightarrow \mathrm{SL}_n(\mathcal{O}); (X, Y) \mapsto e_m^{\mathcal{O}}(X, Y)$.
- ▶ In 2007, Shalev conjectured that the m -th *Engel word* is almost surjective for any non-abelian finite simple group.
- ▶ Bandman, Garion, and Grunewald (2012) showed that the m -th *Engel word* is surjective for the groups $\mathrm{PSL}(2, q)$ and almost surjective for $\mathrm{SL}(2, q)$ provided q is large enough.

Engel maps on SL_n

- ▶ Bandman and Zarhin (2016) showed the surjectivity of *Engel words* for $PSL(2, \mathbb{C})$.
- ▶ Gordeev (2015) studied Engel words on algebraic groups of small ranks.
- ▶ Shalev conjectured the following: Every element of $SL_n(\mathbb{Z}_p)$ where p is a prime and $n \geq 2$ ($p > 3$ when $n = 2$) is a commutator.
- ▶ Ghosh, Meiri, Sarnak (2021) showed that the commutator is not surjective on $SL_2(\mathbb{Z})$.

Engel maps on $SL_2(\mathcal{O})$

- ▶ Roy and S... studied Engel maps over rings.
- ▶ Let $\mathcal{O}$ be a complete, local principal ideal ring and its residue field $k \cong \mathbb{F}_q$ of characteristic $\neq 2$. Consider the induced word maps $e_{m+1}^{\mathcal{O}}$ on $SL_2(\mathcal{O})$ and $\bar{e}_{m+1}$ on $SL_2(k)$.

Let $A \in SL_2(k)$ be a non-scalar element. Then, there exists a constant $q_0(m)$ such that for all $q \geq q_0(m)$, all lifts of A in $SL_2(\mathcal{O})$ are in the image $im(e_{m+1}^{\mathcal{O}})$.

- ▶ Let $\mathcal{O}_2$ be a local principal ideal ring of length 2, with its residue field $k \cong \mathbb{F}_q$ of characteristic $\neq 2$. Then, there exists a constant $q_0(m)$ such that for all $q \geq q_0(m)$, the $(m+1)$ -th Engel word map $e_{m+1}^{\mathcal{O}_2}$ is surjective on $PSL_2(\mathcal{O}_2)$.

Lie algebras

Basics - Lie polynomials

- ▶ Given a set $X = \{x_i \mid i \in I\}$ (we assume it to be finite) we have **free associative algebra** $F(X)$ on X .
- ▶ $F(X)$ is a Lie algebra with the bracket operation

$$[a, b] = ab - ba.$$

- ▶ The **free Lie algebra** $\mathfrak{L}(X)$ on X is the smallest Lie subalgebra of $F(X)$ containing X . The universal enveloping algebra of $\mathfrak{L}(X)$ is $F(X)$.
- ▶ A **Lie polynomial** is an element of a free Lie algebra which defines a map on a given Lie algebra L .
- ▶ One can again ask all the usual questions.

Lie polynomials

- ▶ A Lie monomial in the elements of X is a finite product of elements of X bracketed by Lie brackets in any manner. For example, $[[[[x_1, x_2], x_3], [x_4, x_5]], x_2]$.
- ▶ A Lie word/polynomial is a finite linear combination of Lie monomials.
- ▶ Multilinear polynomial

Engel word

- ▶ Following on the work of Shoda for the Lie algebra of type A_l , Brown (1963) proved the analogue of Ore's conjecture for the classical Lie algebra. That is, the map $[x_1, x_2]$ is surjective with some mild assumption on the base field.
- ▶ In 2012, Bandman, Gordeev, Kunyavskii, and Plotkin proved that the Lie polynomial map (in finitely many words) is dominant for any Chevalley algebra, provided the field is of characteristic $\neq 2$.
- ▶ In the same work, they showed that for the Engel monomial $[[[x, y], y], \dots, y]$ and their linear combinations the map is surjective on the set of non-central elements provided the base field is big enough.

L'vov-Kaplansky type questions

- ▶ Anzis, Emrich, Valiveti (2015) described the images of multilinear Lie polynomials of degree 3 and 4 on certain classical Lie algebras.
- ▶ Kanel-Belov, Malev, and Rowen (2017) describe the images for Lie polynomials on $M_2(\mathbb{K})$.
- ▶ Niranjana Nehra and Shushma Rani (2023) have obtained some results for multilinear Lie polynomials of degree 2 on certain nilpotent Lie algebras.

Nature of the image set

Properties of the image of a word map on groups

- ▶ Let $w = X_{i_1}^{j_1} \cdots X_{i_k}^{j_k}$.
- ▶ The image set $w(G)$ has two properties.
 - (1) $e \in w(G)$.
 - (2) For an automorphism, $\phi \in \text{Aut}(G)$, $\phi(w(G)) \subset w(G)$. This is because,

$$\phi(g_{i_1}^{j_1} \cdots g_{i_k}^{j_k}) = \phi(g_{i_1})^{j_1} \cdots \phi(g_{i_k})^{j_k}.$$

- ▶ In particular, for any $g \in G$ we have $g.w(G).g^{-1} = w(G)$.

Question

- ▶ Given a group G **determine subsets** $A \subset G$ which are possible images, i.e, there exists w with $A = w(G)$.
- ▶ For a given such subset A **determine words** w with the property $A = w(G)$.

Lubotzky's result for FSG

- ▶ For the symmetric group and alternating group, results were obtained by Kassabov and Nikolov.
- ▶ **(Lubotzky 2014)** For any finite simple group G , and a subset A satisfying
 - (a) $e \in A$, and
 - (b) for any $\phi \in \text{Aut}(G)$, $\phi(A) = A$,there exists a word w on two generators such that $w(G) = A$.
- ▶ (Levy 2014) extended this result to quasi-simple groups.

Properties of the image of a Lie map

- ▶ Let L be a (simple) Lie algebra.
- ▶ Let $\text{Aut}(L)$ denote the group of Lie algebra automorphisms for the Lie algebra L .
- ▶ If a subset $A \subset L$ is the image of some Lie polynomial, then it has the following properties:
 1. $0 \in A$.
 2. $\phi(A) = A$, for all $\phi \in \text{Aut}(L)$.
- ▶ **Question:** Let $A \subset L$ satisfying the two above properties. Does there exist a Lie polynomial w in finitely many variables such that $w(L) = A$? If so, can we find some such Lie polynomials that map the set A onto their image?

Some results

- ▶ Bandman et. al. showed dominance of Lie maps on Chevalley Lie algebras.
- ▶ Kanel-belov et. al. showed that over an algebraically closed field of characteristic $\neq 2$, the possible images of Lie polynomials on sl_2 are: the whole set sl_2 , $\{0\}$, or the subset consisting of all non-nilpotent elements.
- ▶ However, there could be many more orbits under the automorphism action, for example, on the Lie algebra $sl_n(q)$, any symmetric $GL_n(q)$ conjugacy class would be such a subset.

Our results (Kishnani, S...)

- ▶ **Theorem:** Let L be a simple Chevalley algebra over a finite field F of “very good” characteristic. Let $k(L)$ denote the minimum number such that for each non-zero element $a \in L$, there exists elements $a_2, a_3, \dots, a_{k(L)}$ in L such that the set $\{a_1 = a, a_2, a_3, \dots, a_{k(L)}\}$ generates L . Then every subset $A \subset L$ containing 0 and closed under automorphisms of L is an image of some Lie polynomial from $\mathcal{F}_{k(L)}$.
- ▶ The number $k(L)$ is a finite number because of dimension reasons, although an absolute constant is desirable.
- ▶ For finite simple groups, it is 2 what is called one-and-a-half generation, but it doesn't seem to be known for all finite simple Lie algebras (over an algebraically closed field, it is known to be 2).

Our results: part 2

- ▶ For the Lie algebra $sl_2(q)$, q odd, we produce Lie polynomials which give a subset as its image.
- ▶ For this we look at $GL_2(q)$ orbits on $sl_2(q)$.
- ▶ For example, the set of all nilpotent elements (including zero) is obtained as the image of the Lie polynomial $e_2(X, Y) - e_{2q}(X, Y)$.
- ▶ The image of the Lie polynomial $e_{q+1}(X, Y) - e_{2q}(X, Y)$ contains all semisimple elements (of both types) together with zero.

Our results: part 2

We consider the following Lie polynomials to get the result.

- ▶ $w_1 = [ad_A^i(X) - ad_A^j(X), [ad_A^{i_1}(X), ad_A^{j_1}(X)]]$
- ▶ $w_n = \left[\cdots \left[w_1, [ad_A^{i_2}(X), ad_A^{j_2}(X)] \right], \cdots, [ad_A^{i_n}(X), ad_A^{j_n}(X)] \right] =$
 $\left[w_{n-1}, [ad_A^{i_n}(X), ad_A^{j_n}(X)] \right]$
- ▶ $w_{1,n} :=$
 $\left[\cdots \left[ad_A^{\alpha_1}(X) - ad_A^{\beta_1}(X), [ad_A^{i_1}(X), ad_A^{j_1}(X)] \right], \cdots, [ad_A^{i_n}(X), ad_A^{j_n}(X)] \right]$
- ▶ for $m \geq 2$, $w_{m,n} := [w_{m-1,n}, ad_A^{\alpha_m}(X) - ad_A^{\beta_m}(X)]$.

Thank You.

email : anupamk18@gmail.com

<https://sites.google.com/site/anupamk182/>