

On Some More Problems from the Kourovka Notebook

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(after joint work with Artem Semidetnov, [arXiv:2608.29219](https://arxiv.org/abs/2608.29219))

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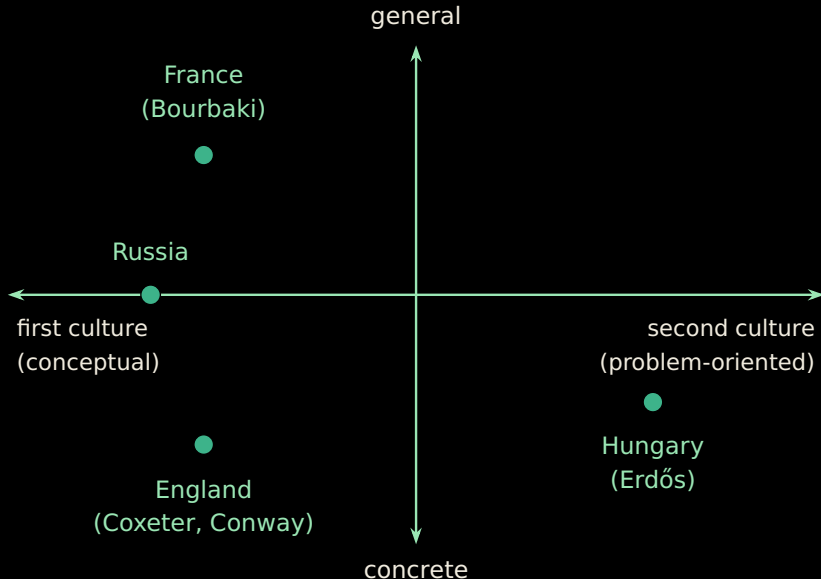
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Chronology (personal story)

- **Second half of 2025:** LLMs begin to produce proofs of nontrivial facts. We use them mainly to write code for CAS, test conjectures, and search the literature.
- **Spring 2026:** New models construct multi-step arguments and devise new tricks (**hard to believe!**).
- **Late spring 2026:** Reports begin to appear of AI systems solving open problems in batches.
- **Summer 2026:** The trend accelerates.
- **July 19, 2026:** L. Alpöge posts a counterexample to the 3D Jacobian conjecture (*the story is not quite so simple*).

The Compass of N. A. Vavilov

(after W. T. Gowers, "The Two Cultures of Mathematics")



Chronology (in parallel)

- **October 2025:** *Aristotle*¹ appears, a system for finding theorem proofs verified in Lean.
- **March 2026:** Harmonic releases Aristotle Agent online. It accepts problem statements in natural language.
- **July 20, 2026:** Four authors submit the preprint *On Some Problems from the Kourovka Notebook*² to arXiv.

¹<https://arxiv.org/abs/2510.01346>

²Wouter van Doorn, Elias Judin, Pietro Monticone, and Daniel Morrison.
<https://arxiv.org/abs/2607.17477>

Overview of results

Q: What did we do?

A: We solved eight open problems in group theory—six from the Kourovka Notebook—with extensive use of AI: Codex with ChatGPT-5.6 Sol and Claude Desktop with Claude Opus 5.

Q: Why did we do it?

A: We explored new possibilities for researchers and experimented with emerging tools in one of our favorite areas of mathematics.

Q: Why is it worth doing?

A:

- Help researchers who have worked on these problems.
- Add verified solutions to the corpus available to future LLMs.
- Develop and compare productive workflows for working with AI.

Problem 1/8

Primitive-preserving endomorphisms

Question (Kourovka 14.85)

Let $M_n = F_n/F_n''$ be the free metabelian group of finite rank n . An element is *primitive* if it belongs to a free basis.

If an endomorphism $\varphi: M_n \rightarrow M_n$ sends every primitive element to a primitive element, must it be an automorphism?³

Answer: Yes

Every primitive-preserving endomorphism of M_n is an automorphism, in every finite rank.

³<https://shpilrain.ccny.cuny.edu/gworld/problems/oproblems.html>

Problem 2/8

Equal distributions in free metabelian groups

Question (Kourovka 19.94)

Do $u, v \in M_n = F_n/F_n''$ have the same distribution on every finite metabelian group if and only if they lie in the same $\text{Aut}(M_n)$ -orbit?

Answer: No

In $M_2 = \langle x, y \rangle$, put $c = [x, y]$ and $a^z = z^{-1}az$. Take

$$\begin{aligned}g_1 &= c^{x^2y^7}(c^x)^{-1}(c^{xy^7})^{-1}c, \\g_2 &= c^{x^3y^7}(c^x)^{-1}(c^{x^2y^7})^{-1}c.\end{aligned}$$

They have equal distributions but distinct automorphism orbits.

Problem 3/8

Derived subgroups as Frattini subgroups

Question (Kourovka 16.11)

For every finite p -group G , does there exist a finite p -group H such that

$$\Phi(H) \cong [G, G]?$$

Here $\Phi(H)$ is the intersection of all maximal subgroups of H .

Answer: No

A counterexample is the finite 2-group

$$G = \text{SmallGroup}(256, 511).$$

Its derived subgroup is $\text{SmallGroup}(32, 32)$, which cannot be the Frattini subgroup of any finite 2-group.

Problem 4/8

Nilpotent Camina groups

Question (Kourovka 17.47)

Let G be a nilpotent group satisfying

$$x^G = xG' \quad \text{for every } x \in G \setminus G',$$

where x^G is the conjugacy class of x .

Is there a universal bound on the nilpotency class of G ?

Answer: No

For every integer $c \geq 2$, there exists a nilpotent Camina group of class exactly c .

The examples arise as groups corresponding to some metabelian nilpotent Lie algebras over $\mathbb{Q}$.

Problem 5/8

Minimal T -systems and negative immersions

Question (Linton–Nyberg-Brodda, 2.8.14)

Let X be a one-relator complex with negative immersions. Does $\pi_1(X)$ have a unique T -system of generating sets of minimal cardinality?

A T -system is an equivalence class of generating tuples under Nielsen moves and group automorphisms.

Answer: No

For every $N \geq 1$, there is a three-generated, 2-free one-relator group with at least N distinct T -systems of minimal generating triples.

Each triple comes from a one-relator presentation whose natural complex has uniform negative immersions.

Problem 6/8

Word-defined group laws

Question (Bardakov et al., Problem 18)

Let $w = w(a, b) \in F_2$ be a reduced word. Suppose that

$$x *_w y = w(x, y)$$

defines a group operation on $F_n/\gamma_k(F_n)$ for all $n \geq 2$ and $k \geq 1$, where γ_k is the lower central series.

Must $w = ab$ or $w = ba$?

Answer: Yes

The only possibilities are

$$w(a, b) = ab \quad \text{or} \quad w(a, b) = ba,$$

giving the usual multiplication and the opposite multiplication.

Problem 7/8

Stable representations of nilpotent groups

Question (Plotkin, Problem 4.3.5)

Let $n \geq 2$, let G be a torsion-free nilpotent group of class $n - 1$, and let $H \trianglelefteq G$ be abelian.

Does G admit a faithful n -step stable representation whose restriction to H is 2-step stable?

Answer: Yes, over $\mathbb{Q}$

There exists a faithful representation on a rational vector space M satisfying

$$M\Delta_G^n = 0, \quad M\Delta_H^2 = 0.$$

Here Δ_G and Δ_H are the augmentation ideals of $\mathbb{Q}G$ and $\mathbb{Q}H$, respectively.

Problem 8/8

A Cayley–Hamilton analogue for free groups

Question (Kourovka 17.32)

Let $w \in F_n$ and $\varphi \in \text{Aut}(F_n)$. Does

$$\langle w, \varphi(w), \dots, \varphi^n(w) \rangle = F_n$$

imply $\langle w, \varphi(w), \dots, \varphi^{n-1}(w) \rangle = F_n$?

Answer: No, already in rank two

In $F_2 = \langle x, y \rangle$, take

$$w = xy, \quad \varphi(x) = yx, \quad \varphi(y) = x.$$

Then $\varphi(w) = yx^2$ and $\varphi^2(w) = w^2x$, so

$$\langle w, \varphi(w), \varphi^2(w) \rangle = F_2, \quad \langle w, \varphi(w) \rangle \neq F_2.$$

Thank you for your attention

