

# Saint Petersburg, Nikolay Vavilov's memorial conference

## Group schemes over LG-rings and Applications

Philippe Gille  
Institut Camille Jordan, Lyon

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# Outline

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- 2 Maximal tori
- 3 Parabolic subgroups
- 4 Minimal parabolics
- 5 Cancellation
- 6 Azumaya algebras

LG rings were discovered in the 80's and came back recently in the book of Garibaldi-Petersson-Racine *Albert Algebras over Commutative Rings: The Last Frontier of Jordan Systems*.

The goal today is to extend several results from SGA3 from semilocal rings to the larger class of **LG-rings**.

The main applications concern:

- maximal tori;
- parabolic subgroups;
- minimal parabolic subgroups;
- maximal split subtori;
- Group of points (à la Vavilov);
- Cancellation theorems;
- Azumaya algebras.

# Definition of an LG-ring

Let

$$f \in R[X_1, \dots, X_n].$$

We say that  $f$  represents a unit over  $R$  if there are  $a_1, \dots, a_n \in R$  such that

$$f(a_1, \dots, a_n) \in R^\times.$$

## Definition

A ring  $R$  is an LG-ring if, for every polynomial  $f$ ,  $f$  represents a unit over  $R$  if and only if  $f$  represents a unit over every residue field

$$R/\mathfrak{m}, \quad \mathfrak{m} \in \text{Max}(R).$$

- The condition on residue fields is not easy in the finite field case.
- For example, the polynomial  $X^p - X$  provides the zero function on  $\mathbb{F}_p$ .

Important examples include:

- fields;
- semilocal rings;
- zero-dimensional rings;
- the ring of algebraic integers  $\overline{\mathbb{Z}}$  (Dade, 1963).
- A finite product of LG-rings is LG and any integral extension of a LG ring is LG.
- In particular, every finite  $R$ -algebra of an LG-ring  $R$  is itself an LG-ring.
- On the other hand,  $\mathbb{Z}$  (and  $\mathbb{C}[t]$ ) is not an LG-ring.
- For example, the polynomial  $2X^2 + 3Y^2$  does not represent  $\pm 1$  over  $\mathbb{Z}$ .

- We say that a LG-ring  $R$  is LGG if all residue fields are infinite.
- In this case, the condition on residue fields above for a polynomial  $P$  is to require that the ideal of  $R$  generated by coefficients of  $P$  is  $R$ . We say that  $P$  is primitive.
- The LGG condition is then that for every primitive polynomial  $P \in R[X]$  there exists  $r \in R$  such that  $P(r) \in R^\times$ .
- The literature says that  $R$  satisfies the primitive criterion;
- $\overline{\mathbb{Z}}$  is a LGG ring.

For an LG-ring  $R$ , we have the following result of Estes-Guralnick (1982):

$$M \cong N$$

for finitely presented modules  $M, N$  if and only if their localizations at every maximal ideal are isomorphic.

- In particular, every finite projective module of constant rank is free.
- Therefore

$$\mathrm{Pic}(R) = 0.$$

- We shall come back on that.

# Geometric interpretation

Let  $M$  be a finite locally free  $R$ -module. The notation

$$\mathbf{W}(M)$$

denotes the affine  $R$ -scheme associated with  $M$  (called also the vector group scheme associated to  $M$ ).

## Proposition (★)

Let  $R$  be an LG-ring and let

$$U \subset \mathbf{W}(M)$$

be open and quasi-compact. Then

$$U(R) \neq \emptyset$$

if and only if

$$U(R/\mathfrak{m}) \neq \emptyset$$

for every maximal ideal  $\mathfrak{m}$  of  $R$ .

Let  $M$  be a locally free  $R$ -module of rank  $n \geq 1$  where  $R$  is LG. We consider the scheme of "unimodular vectors"  $U = W(M) \setminus \{0\}$ .

- Proposition  $(\star)$  shows that  $U(R) \neq \emptyset$ , that is,  $M$  admits an unimodular element  $m$ .
- Then  $Rm$  is a direct summand of  $M$  and by induction on the rank,  $M$  is a free  $R$ -module.

# Why Proposition $(\star)$ is relevant

Many algebraic questions can be reformulated as finding a point in an open subscheme of vector group scheme. Thus the typical strategy is:

construct an open quasi-compact subscheme of a vector group scheme



solve the problem over every residue field



apply Proposition  $(\star)$



obtain a global solution.

# Existence of maximal tori

Let  $G$  be a reductive group scheme over an LG-ring  $R$ .

The paper proves:

## Theorem

Every parabolic subgroup of  $G$  contains a maximal  $R$ -torus.

In particular,  $G$  itself contains a maximal  $R$ -torus.

This extends the corresponding result of SGA3 from semilocal rings to LG-rings.

The proof follows the philosophy of SGA3.

- One constructs an appropriate parameter space for maximal tori.
- The desired maximal torus exists over every residue field by the theory of algebraic groups.
- The relevant parameter space is sufficiently affine and open.
- Therefore Proposition  $(\star)$  gives a global point and a maximal  $R$ -torus of  $G$ .

We can reduce to the case  $G$  adjoint.

- We consider the open quasi-compact subscheme  $U$  of  $W(\mathrm{Lie}(G))$  consisting in the  $X \in \mathrm{Lie}(G)$  which are (semisimple) regular, that is, which are fiberwise regular.
- In other words,  $U(R)$  consists in the  $X \in \mathrm{Lie}(G)(R)$  such that for each (maximal) ideal  $\mathfrak{m}$  of  $R$ , the centralizer  $C_{\mathrm{Lie}(G)_{R/\mathfrak{m}}}(X_{R/\mathfrak{m}})$  is a Cartan subalgebra.
- If  $R/\mathfrak{m}$  is infinite, it is well-known that  $U(R_{\mathfrak{m}}) \neq \emptyset$ . In the finite field case, this is a result of Chevalley and Serre.
- It follows that  $U(R)$  is not empty and taking the centralizer provides a Cartan  $R$ -subalgebra of  $\mathrm{Lie}(G)$  whence a maximal  $R$ -torus of  $G$ .

# Parabolic subgroups

Let  $G$  be a reductive  $R$ -group schemes.

- We denote by  $\text{Par}(G)$  the (projective)  $R$ -schemes of parabolic subgroups of  $G$ . Its Stein factorization provides a type map

$$\text{type} : \text{Par}(G) \rightarrow \text{Of}(\text{Dyn}(G))$$

- For example, if  $G$  is split with Dynkin diagram  $\Delta_0$  and  $R$  is connected, the type of a parabolic  $R$ -subgroup  $P$  of  $G$  is a subset of  $\Delta_0$ .
- The main result is:

## Theorem

Assume that  $R$  is an LG-ring. Two parabolic subgroups of the same type of  $G$  are conjugate by an element of  $G(R)$ .

# The big-cell picture

The proof involves the big cell. We continue with our LG-ring  $R$ , a reductive  $R$ -group  $G$  equipped with a pair of opposite  $R$ -parabolic subgroup  $P = U \rtimes L$ ,  $P' = U' \rtimes L$ .

- One proves a decomposition of the form

$$G(R) = U(R) U'(R) P(R)$$

by using the big cell  $U'$  in  $G/P$ .

- The important point is that the relevant parameters lie in unipotent groups, and unipotent groups are affine spaces (as  $R$ -schemes).

## Case of a split Chevalley $R$ -group

We deal now with a Chevalley semisimple simply connected  $R$ -group scheme  $G_0$  equipped with opposite Borel subgroups  $B_0 = U_0 \rtimes T_0$ ,  $B'_0 = U'_0 \rtimes T_0$ .

- We have

$$G_0(R) = U_0(R) U'_0(R) B_0(R) = U_0(R) U'_0(R) U_0(R) T_0(R).$$

- It implies that  $G_0(R)$  is generated by  $U_0(R)$  and  $U'_0(R)$ .
- It is the kind of result in the circle of Vavilov's work.
- The proof is immediate from the decomposition, we need to deal with the elements of  $T_0(R)$ . It boils down to the  $\mathbf{SL}_2$  case handled by the formula

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

# Cohomological consequence

The conjugacy theorem has an important consequence.

- For  $G, P$  as before and  $R$  a LG-ring, the map

$$H^1(R, P) \longrightarrow H^1(R, G) \text{ is injective.}$$

- Here  $H^1$  denotes fppf (or étale) cohomology and classify torsors. In other words a  $P$ -torsor which becomes trivial as a  $G$ -torsor was already trivial.
- Indeed we have an exact sequence of pointed sets

$$1 \longrightarrow P(R) \longrightarrow G(R) \longrightarrow (G/P)(R) \longrightarrow H^1(R, P) \longrightarrow H^1(R, G)$$

and  $(G/P)(R)$  is nothing but the set of  $R$ -parabolic subgroups of same type than  $P$  (since  $P = N_G(P)$ ).

# Minimal parabolic subgroups

Assume from now on that  $R$  is connected.

- A parabolic subgroup is minimal if it contains no proper parabolic subgroup.
- Since parabolics are smooth of constant dimension, minimal parabolics exist.
- More importantly:

## Theorem

$G(R)$  acts transitively on the set of minimal parabolic subgroups of  $G$ .

- Thus any two minimal parabolic subgroups are conjugate over  $R$ .

# Maximal split subtori

Suppose furthermore that  $G$  is semisimple. The previous theorem also gives:

## Theorem

$G(R)$  acts transitively on the maximal split subtori of  $G$ .

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- Therefore maximal split subtori are unique up to  $G(R)$ -conjugacy.
- This generalizes the familiar result over fields.
- The proof involves the parameterization of parabolic subgroups by  $\lambda : \mathbb{G}_m \rightarrow G$ , i.e. Richardson's parabolic subgroups  $P_G(\lambda)$ .

# The cancellation principle

We deal now with the following picture

$$\begin{array}{ccccc} L & \hookrightarrow & P & \hookrightarrow & H \\ & \downarrow & & & \\ & G & & & \end{array}$$

where:

- $P$  is parabolic in  $H$ ;
- $L$  is a Levi subgroup of  $P$ ;
- the map  $L \rightarrow G$  admits a section.
- Then

$$H^1(R, G) \longrightarrow H^1(R, H) \text{ is injective.}$$

- This is the basic cohomological cancellation principle which will be applied in various situations.

# Cancellation of projective modules

For finite faithfully projective modules:

$$M \otimes_R N \cong M' \otimes_R N$$

implies

$$M \cong M'.$$

In particular,

$$M^{\oplus n} \cong M'^{\oplus n} \implies M \cong M'.$$

The relevant groups are general linear groups. The stabilization corresponds to passing from  $\mathbf{GL}_d$  to a suitable Levi subgroup of  $\mathbf{GL}_{d+e}$ .

# Cancellation of Azumaya algebras

Let  $A, B, C$  be Azumaya algebras over an LG-ring (respectively of constant degrees  $d_A = d_B, d_C$ ).

- The cohomological cancellation yields:

$$A \otimes_R C \cong B \otimes_R C \implies A \cong B.$$

- It reduces to the case  $C$  is split since  $C \otimes_R C^{op} \cong \text{End}_R(C) \cong M_{d_C^2}(R)$ . We put  $n = d_C^2$ .
- For this case,  $H = \text{PGL}_n(A) = \text{GL}_n(A)/\mathbb{G}_m$ ,  $L = \text{GL}_1(A)^n/\mathbb{G}_m$  and  $G = \text{PGL}_1(A)$ . The morphism  $L \rightarrow G$  is the first projection and split by the diagonal.
- The resulting map  $H^1(R, \text{PGL}_1(A)) \rightarrow H^1(R, \text{PGL}_n(A))$  applies  $[B]$  to  $[M_n(B)]$ .
- This is stronger than cancellation in the Brauer group.

Let  $S/R$  be a quadratic étale extension with involution.  
For hermitian forms:

$$h_1 \perp h_3 \cong h_2 \perp h_3$$

implies

$$\boxed{h_1 \cong h_2.}$$

The proof uses unitary group schemes and the cohomological cancellation principle.

# Quadratic forms

For regular quadratic forms:

$$q_1 \perp q_3 \cong q_2 \perp q_3$$

implies

$$\boxed{q_1 \cong q_2.}$$

This is the Witt cancellation theorem over connected LG-rings.  
The proof uses special orthogonal group schemes

$$\mathbf{SO}(q)$$

together with the relationship between  $\mathbf{SO}(q)$  and  $O(q)$ .

Let  $A$  be an Azumaya algebra over a connected LG-ring  $R$ .

The paper studies finite projective  $A$ -modules and the Brauer class

$$[A] \in \mathrm{Br}(R).$$

The central result is a uniqueness statement for indecomposable objects.

# Indecomposable finite projective modules

A finite projective  $A$ -module  $P$  is indecomposable if

$$\text{idemp}(\text{End}_A(P)) = \{0, 1\}.$$

The paper proves:

## Theorem

Any two indecomposable finite projective  $A$ -modules are isomorphic.

This is one of the main applications of the LG local-global principle.

# Idea of the proof

- Let  $P, Q$  be indecomposable finite projective  $A$ -modules.
- Consider

$$U = \{f \in \operatorname{Hom}_A(P, Q) : f \text{ is surjective}\}.$$

- Surjectivity is an open condition. Over every residue field, the corresponding open subset has a point.
- Hence Proposition  $(\star)$  gives

$$U(R) \neq \emptyset.$$

- Therefore there is a surjection

$$P \twoheadrightarrow Q.$$

- Since  $Q$  is projective, this splits.

# The Wedderburn property

The preceding theorem implies that every Brauer class contains, up to isomorphism, a unique algebra  $B$  satisfying

$$\text{idemp}(B) = \{0, 1\}.$$

Thus connected LG-rings have the

Wedderburn property.

This generalizes the uniqueness of the division algebra representative of a Brauer class over a field.

# Brauer decomposition

Suppose that the index of a Brauer class has prime factorization

$$\mathrm{ind}([A]) = p_1^{n_1} \cdots p_r^{n_r}.$$

The paper obtains a decomposition

$$A \cong B_1 \otimes_R \cdots \otimes_R B_r$$

where each  $B_i$  is indecomposable and has degree a power of the corresponding prime  $p_i$ .  
The decomposition is unique up to isomorphism.

# What about exceptional groups?

The above results were for classical group schemes. In the exceptional case, let me state a few results by Garibaldi-Petersson-Racine.

- An octonion  $R$ -algebra is a Cayley-Dickson construction.
- It is determined by its norm.
- There are results on Albert algebras, read the book!



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- Thanks for attention.

Спасибо !

