

Reductive Group Schemes over LG-Rings and Applications

Philippe Gille

Abstract. This talk is joint work with Erhard Neher and is based on the paper published in the Nikolai Vavilov Memorial Issue of the European Journal of Mathematics [G-N]. We study reductive group schemes over local-global (LG) rings and develop a local-global approach to their structure. It applies to conjugacy results for parabolic subgroups and for maximal split subtori.

Introduction and Motivation

The foundational treatise SGA3 by Demazure-Grothendieck establishes the theory of reductive group schemes over a base ring R . Several central results—existence of maximal tori, conjugacy of parabolic subgroups, and transitivity properties—are, however, only proved in the special case where R is a *semilocal* ring. Our purpose is to extend most of these results to the strictly larger class of *LG-rings* (local-global rings).

LG-rings. The concept of an LG-ring, introduced by McDonald and Waterhouse [M-W] and first studied in depth by Estes and Guralnick [E-G], axiomatizes a key property of semilocal rings: a polynomial in several variables represents a unit of R if and only if it represents a unit over every residue field $R/\mathfrak{m}$ (equivalently, over every localization $R_{\mathfrak{m}}$). This “local-global” principle for units is the origin of the name. The class of LG-rings properly contains:

- all semilocal rings (hence all fields and all artinian rings);
- rings that are von Neumann regular modulo their Jacobson radical;
- zero-dimensional rings;
- the ring of all algebraic integers (and all real algebraic integers);
- any finite R -algebra or integral extension of an LG-ring.

Notable non-examples include $\mathbb{Z}$ and polynomial rings $R[X]$ over an integral domain.

Motivation. The subject of LG-rings fell into near-oblivion until it was revived by Garibaldi, Petersson, and Racine [G-P-R1, G-P-R2] in their work on quadratic forms and Jordan algebras. The present talk provides a geometric approach to LG-rings and systematically extends the SGA3 toolkit, opening the way to applications in the theory of forms and Azumaya algebras.

1. Geometric Characterization of LG-Rings

The key technical innovation is a geometric reformulation of the LG-property that enables the transfer of SGA3 arguments from the semilocal to the LG setting.

Geometric characterization. *Let R be an LG-ring, M a finite locally free R -module, and $U \subset \mathbf{W}(M)$ an open quasi-compact subscheme of the affine scheme $\mathbf{W}(M) = \mathrm{Spec}(\mathrm{Sym}(M^*))$ associated with M . Then:*

$$U(R) \neq \emptyset \iff U(R/\mathfrak{m}) \neq \emptyset \text{ for every maximal ideal } \mathfrak{m} \triangleleft R.$$

This local-to-global lifting of rational points is the engine of the paper. The proof leverages the fact that for a quasi-compact open U defined by a finitely generated ideal I , one can construct an auxiliary linear polynomial that represents a unit modulo every $\mathfrak{m}$; the LG-property then promotes it to a unit over R . A companion result handles the *primitive condition*—rings that are LG with all residue fields infinite—where R -density of U suffices.

A first application is a streamlined proof that finite projective R -modules of constant rank over an LG-ring are free, recovering $\mathrm{Pic}(R) = 0$ by an inductive argument using unimodular elements.

2. Existence of Maximal Tori

It generalizes Grothendieck’s theorem on the existence of maximal tori in reductive group schemes [SGA3, XIV, 3.20] from semilocal rings to LG-rings. The proof follows the SGA3 approach via regular elements of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)(R)$. An important step is

Let G be a reductive group scheme over an LG-ring R satisfying the primitive condition, or let G be adjoint. Then $\mathfrak{g}$ admits a regular element.

The main result is the following:

Let G be a reductive group scheme over an LG-ring R . Then G admits a maximal R -torus.

A further issue is that every parabolic subgroup admits a maximal torus.

3. Parabolic Subgroups: Conjugacy and the Tits Index

We come now to the conjugacy theorems for parabolic subgroups.

Let G be a reductive group scheme over an LG-ring R . Two parabolic subgroups of G of the same type are conjugate under $G(R)$.

This generalizes the classical semilocal result. The proof again uses the geometric characterization of LG rings: the relevant quotient scheme has $R/\mathfrak{m}$ -points for all $\mathfrak{m}$, hence an R -point.

It provides a unique smallest type $\mathbf{t}_{\min}$ in the set of types of parabolic subgroups of G over an LG-ring. When G is semisimple and R is connected, allowing to define the *Tits index* of G over R . It turns out that no new indices arise beyond those already known from the field case.

4. Minimal Parabolics and Maximal Split Tori

We focus now on minimal parabolic subgroups, their Levi subgroups, and maximal split tori in reductive group schemes over a *connected* LG-ring.

Let R be a connected LG-ring and G a reductive R -group scheme.

- (a) *$G(R)$ acts transitively on the set of minimal parabolic subgroups of G .*
- (b) *If G is semisimple, $G(R)$ acts transitively on the maximal split R -subtori of G .*

This transitivity is the analogue of the Borel–Tits theory over fields and is essential for defining structural invariants. As a consequence, one defines the *anisotropic kernel* of G as the derived group of a minimal Levi subgroup—well-defined thanks to the conjugacy result. This invariant plays a role analogous to the anisotropic kernel in the classical theory of algebraic groups over fields.

5. Application : Cancellation Theorems

It leads to cancellation theorems via a cohomological principle having the following consequences:

- **Cancellation of modules and Azumaya algebras in tensor products:** If $P \otimes Q_1 \cong P \otimes Q_2$ for suitable finite projective modules or Azumaya algebras, then $Q_1 \cong Q_2$.
- **Cancellation of hermitian forms:** analogues of Witt cancellation for hermitian forms over LG-rings, extending results from the semilocal case.
- **Witt cancellation of quadratic forms :** if $q_1 \perp q_3 \cong q_2 \perp q_3$ for quadratic forms q_1, q_2, q_3 over an LG-ring, then $q_1 \cong q_2$. This generalizes the earlier result of Dias [D] from her Ph.D. thesis, which was limited to a more restrictive class of rings.

These cancellation theorems are the algebraic-geometric counterparts of the classical cancellation results for forms over fields and semilocal rings, now valid over the broader LG setting.

Conclusion

We extended important results from fields and semilocal rings to the broader class of LG-rings. In particular one obtain cancellation theorems for projective modules and Azumaya algebras. They also establish cancellation results for hermitian and quadratic forms over LG-rings.

Overall, the work demonstrates the strength of LG-rings as a natural setting for generalizing classical algebraic structures and cancellation phenomena.

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Philippe Gille
 Institut Camille Jordan - Université Claude Bernard Lyon 1
 43 boulevard du 11 novembre 1918,
 69622 Villeurbanne cedex - France
 e-mail: gille@math.univ-lyon1.fr