

Inducing graph invariants from the universal $\mathfrak{gl}$ -weight system

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Abstract. Weight systems, which are functions on chord diagrams satisfying certain 4-term relations, appear naturally in Vassiliev's theory of finite type knot invariants. In particular, a weight system can be constructed from any finite dimensional Lie algebra endowed with a nondegenerate invariant bilinear form [5]. Recently, M. Kazarian suggested to extend the $\mathfrak{gl}(N)$ -weight system from chord diagrams (treated as involutions without fixed point) to arbitrary permutations, which led to a recurrence formula allowing for an effective computation of its values, elaborated by Zhuoke Yang [6] (see also [1]). In turn, the recurrence helped to unify the $\mathfrak{gl}(N)$ weight systems, for $N = 1, 2, 3, \dots$, into a universal $\mathfrak{gl}$ -weight system. The latter takes values in the ring of polynomials $\mathbb{C}[N][C_1, C_2, \dots]$ in infinitely many variables $C_1, C_2, \dots$ (Casimir elements), whose coefficients are polynomials in N .

The universal $\mathfrak{gl}$ -weight system carries a lot of information about chord diagrams and intersection graphs. The talk will address the question which graph invariants can be extracted from it. We will discuss the interlace polynomial, the enhanced skew-characteristic polynomial, and the chromatic polynomial. In particular, we show that the interlace polynomial of the intersection graphs can be obtained by a specific substitution for the variables $N, C_1, C_2, \dots$. This allows one to extend it from chord diagrams to arbitrary permutations.

Questions concerning other graph and delta-matroid invariants and their presumable extensions will be formulated.

The talk is based on a work of the speaker, M. Kazarian and a PhD student Nadezhda Kodaneva [2, 3, 4].

References

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