

Automorphic forms, cohomology of arithmetic groups, and moduli spaces

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Abstract. The talk will review results based on the substantive connection between the structure of generators of arithmetic groups and automorphic forms, which allows us to prove the nontriviality of higher cohomology groups.

Introduction

My work is devoted to the theory of automorphic forms, but in these notes I try to present it from the point of view of the theory of arithmetic groups. We are talking about non-triviality of the third cohomology groups of the paramodular (symplectic) groups of genus 2 and the fourth cohomology of some integral orthogonal groups of signature (2,4). Both questions are associated with the irrationality (non-negativity of the Kodaira dimension) of moduli spaces of complex varieties. In our topic, the question of special generators of orthogonal groups is important. For example, we discuss conditions under which Eichler transvections generate a stable integral orthogonal group of an indefinite quadratic lattice over the ring of integers. We will also formulate some questions in this direction.

1. The paramodular group

The *paramodular group* Γ_t ($t \in \mathbb{N}$) is the integral symplectic group of rank 2 of a symplectic form with elementary divisors $(1, t)$. It is the modular group of the moduli space $\mathcal{A}_t$ of polarized abelian surfaces with a polarisation of type $(1, t)$.

We start with natural questions related to the first and second cohomology groups of Γ_t . We note that $\Gamma_1 = Sp_4(\mathbb{Z})$ is perfect. In [2] we analysed the commutator of Γ_t for $t > 1$. We proved that

$$\Gamma_t^{ab} = \Gamma_t / [\Gamma_t, \Gamma_t] = (\mathbb{Z}/(t, 12)\mathbb{Z}) \times (\mathbb{Z}/(2t, 12)\mathbb{Z}).$$

The paramodular group Γ_t is not maximal discrete subgroup of $Sp_4(\mathbb{R})$. Its maximal normal extension is generated by special quasi-involutions (see [2])

$$\Gamma_t^* = \langle \Gamma_t, V_d \rangle, \quad V_d^2 \in \Gamma_t, \quad d|t, \quad (d, \frac{t}{d}) = 1,$$

and $\Gamma_t^*/\Gamma_t \cong (\mathbb{Z}/2\mathbb{Z})^{\rho(t)}$, where $\rho(t)$ is the number of prime divisors of t .

The situation with the third cohomology group of Γ_t is rather non-trivial. In [1] we proved that

$$\text{rank } H^3(\Gamma_t, \mathbb{C}) > 0 \quad \text{if } t \neq 1, \dots, 12, 14, 15, 16, 18, 20, 24, 30, 36.$$

More exactly, in [1] we constructed canonical differential forms on any smooth compactification of the moduli space of $(1, t)$ -polarised abelian surfaces using author's lifting construction of modular forms of weight 3 with respect to Γ_t by Jacobi cusp forms of weight 3 and index t . In other words, our method is transcendental not algebraic. For example, a non-trivial class in $H^3(\Gamma_{13}, \mathbb{C})$ for the first possible $t = 13$ is given by the lifting of the following theta-block

$$\varphi_{3,13}(\tau, z) = \frac{\vartheta(\tau, z)^5 \vartheta(\tau, 2z)^3 \vartheta(\tau, 3z)}{\eta(z)^3} \in J_{3,13}^{cusp},$$

where $\vartheta(\tau, z) = -\vartheta(\tau, -z)$ is the odd Jacobi theta-series of characteristic 2 and $\eta(z)$ is the Dedekind eta-function. (See [7] for the theory of theta-blocks.)

Problem. *For what natural t the group $H^3(\Gamma_t^*, \mathbb{C})$ is not trivial?*

For a prime p the group Γ_p^* is the modular group of the moduli spaces of $(1, p)$ -polarized Kummer surfaces (see [2]). We found the first primes with non-trivial $H^3(\Gamma_t^*, \mathbb{C})$ in [6]. They are $p = 167, 173, 223$, etc.

2. Orthogonal group of $O'(L)$

Here L is an integral even lattice, i.e. a free $\mathbb{Z}$ -module with a non-degenerate bilinear form $(\cdot, \cdot): L \times L \rightarrow \mathbb{Z}$ such that $(u, u) = u^2 \in 2\mathbb{Z}$ for any $u \in L$. The dual lattice L^* contains L . We denote its *discriminant group* by $D(L) = L^*/L$. It carries a finite quadratic forms with values in $\mathbb{Q}/2\mathbb{Z}$. The *stable orthogonal group* $\widetilde{O}(L)$ is defined as the kernel of the natural projection to the finite orthogonal group $O(D(L))$. We consider the spinor norm $\text{sn}_K: O(L \otimes K) \rightarrow K^\times / (K^\times)^2$ over a field K and define the *stable orthogonal group* and the *spinor kernel*

$$\widetilde{SO}^+(L) = \widetilde{SO}(L) \cap \ker \text{sn}_{\mathbb{R}}, \quad O'(L) = SO(L) \cap \ker \text{sn}_{\mathbb{Q}}.$$

We tell that an even integral lattice L satisfies the *Kneser condition* if its Witt index ≥ 2 over $\mathbb{R}$, L represents -2 and that $\text{rank}_3(L) \geq 5$ and $\text{rank}_2(L) \geq 6$.

Theorem. (See [4, §1].) *If L satisfies the Kneser condition then $O'(L) = \widetilde{SO}^+(L)$. Moreover, the abelianisation $\widetilde{O}^+(L)^{ab}$ is an abelian 2-group. Its order divides 2^N , where N is the number of different $\widetilde{O}^+(L)$ -orbits of (-2) -vectors in L . The same is true if we replace $\widetilde{O}^+(L)$ by $\widetilde{SO}^+(L)$ throughout.*

To proof the theorem we used the strong approximation for indefinite orthogonal groups and some considerations with reflections. We got a better result using special unipotent elements called Eichler transvections

$$t(e, a): v \mapsto v - (a, v)e + (e, v)a - \frac{1}{2}(a, a)(e, v)e \in \widetilde{SO}^+(L).$$

The *divisor* $\text{div}(l)$ of $l \in L$ is the positive generator of the ideal $(l, L) \subset \mathbb{Z}$. This is a divisor of $|L^*/L| = |\det(L)|$.

We define $E(L)$ to be the group generated by all transvections by unimodular isotropic vectors:

$$E(L) := \langle t(e, a) \mid e, a \in L, (e, e) = (e, a) = 0, \text{div}(e) = 1 \rangle.$$

Theorem. (See [4, Theorem 1.7].) *Let $L = U \oplus U \oplus L_0$ be an even lattice with two hyperbolic planes U , such that $\text{rank}_3(L) \geq 5$ and $\text{rank}_2(L) \geq 6$. Then*

$$\widetilde{SO}^+(L) = O'(L) = E(L),$$

$\widetilde{SO}^+(L)^{ab}$ is trivial and $\widetilde{O}^+(L)^{ab} \cong \mathbb{Z}/2\mathbb{Z}$.

From this algebraic result the following geometric consequences follow:

Theorem. (See [4, Theorem 5.5 and Theorem 6.7].) *Any smooth model of a compactification of the moduli space $\mathcal{A}_t$ of $(1, t)$ -polarized abelian surfaces is simply-connected. The same is true for the moduli space of $2d$ -polarized K3 surfaces.*

3. $H^4(\Gamma_{t,s}, \mathbb{C})$ where $\Gamma_{t,s} \subset O(2, 4)$

The projectivised paramodular group $\Gamma_t/\{\pm E_4\}$ is isomorphic to the stable special orthogonal group $\widetilde{SO}^+(L_t)$ of the integral lattice $\Lambda_{2t} = U \oplus U \oplus \langle -2t \rangle$ of signature $(2, 3)$ (see [2]). Constructing canonical differential forms (or nontrivial cohomology classes) for the stable orthogonal group of $\widetilde{O}^+(L)$ is much more difficult. The case of signature $(2, 19)$, corresponding to polarized K3 surfaces, was studied in [3]. In the theory of moduli spaces of polarized hyperkähler manifolds, orthogonal groups of signature $(2, 4)$, $(2, 5)$, $(2, 20)$, and $(2, 21)$ arise. See [5], which provides a review of the authors' results for the cases of signatures $(2, 19)$, $(2, 20)$, and $(2, 21)$. Four-dimensional moduli spaces of polarized generalized Kummer manifolds (the case of signature $(2, 4)$) are perhaps the most interesting to date.

We put

$$L_{2d, 2n+2} = U \oplus U \oplus \langle -2d \rangle \oplus \langle -(2n+2) \rangle, \quad \text{sign } L_{2d, 2n+2} = (2, 4).$$

We denote by $v_{-(2n+2)}$ the generator vector of the lattice $\langle -(2n+2) \rangle$ and by $\sigma_{-(2n+2)} \in O^+(L_{2d, 2n+2})$ the reflection defined by the vector $v_{-(2n+2)}$. We consider the double extension of the stable special orthogonal group

$$\Gamma_{2d, 2n+2} = \langle \widetilde{SO}^+(L_{2d, 2n+2}), \sigma_{-(2n+2)} \rangle.$$

Question. The lattice $L_{2d, 2n+2}$ does not satisfy the Kneser conditions from §3. Calculate the abelianization of $\Gamma_{2d, 2n+2}$.

One can show that the moduli space of generalized Kummer varieties of type $\text{Kum}^n(A)$ with a split polarisation of degree $2d$ is a four dimensional modular variety

$$\mathcal{M}(\text{Kum}^n, h_{2d}^{\text{split}}) = \Gamma_{2d, 2n+2} \backslash \mathcal{D}(L_{2d, 2n+2}),$$

where $\mathcal{D}(L_{2d,2n+2})$ is the homogeneous domain of type IV and dimension 4, associated with $L_{2d,2n+2}$. Using the theory of theta-blocks and the author lifting construction from [1] we prove

Theorem. (See [8].)

$$h^{4,0}(\mathcal{M}(\text{Kum}^2, h_{2d}^{\text{split}})) > 0, \quad \text{if } 2d \neq 2, 4, 6, 10, 12, 18;$$

$$h^{4,0}(\mathcal{M}(\text{Kum}^n, h_6^{\text{split}})) > 0, \quad \text{if } n \neq 2, 4, 5, 8.$$

In particular, in these cases the fourth cohomology group of $\Gamma_{2d,2n+2}$ is not trivial.

In [8] we give a much more general result for a very wide set of pairs of dimensions $2n$ and polarisations $2d$.

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