

Automorphic forms, cohomology of arithmetic groups, and moduli spaces

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1. Objects and Questions

1) The *paramodular group* $\Gamma_t \subset Sp_4(\mathbb{Q})$.

$\mathcal{A}_t = \Gamma_t \backslash \mathbb{H}_2$ is the moduli space $\mathcal{A}_t$ of polarized abelian surfaces with a polarisation of type $(1, t)$.

$$H^3(\Gamma_t, \mathbb{C}) \neq \{0\} ?$$

This is a classical question, related to a Siegel problem on the geometric type of these moduli spaces. I proposed a solution in 1993 using a Jacobi lifting construction.

2) A new question we have about the moduli space of $2d$ -polarized *generalised Kummer varieties of type* Kum^n . It is the second known infinite series of hyperkähler manifolds. An arithmetic group $\Gamma_{d,n} \subset O(2, 4)$ is the modular group of this moduli space.

$$H^4(\Gamma_{d,n}, \mathbb{C}) \neq \{0\} ?$$

2. Commutator of the paramodular group Γ_t

$$Sp(J_t, \mathbb{Z}) = \{g \in M_4(\mathbb{Z}) : g^t J_t g = J_t\}, \quad J_t = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$$

is conjugate to

$$\Gamma_t = \left\{ g \in Sp(4, \mathbb{Q}) : g \in \begin{pmatrix} * & t* & * & * \\ * & * & * & */t \\ * & t* & * & * \\ t* & t* & t* & * \end{pmatrix}, * \in \mathbb{Z} \right\}.$$

In [GH98] we proved that

$$\Gamma_t^{ab} = \Gamma_t / [\Gamma_t, \Gamma_t] = (\mathbb{Z}/(t, 12)\mathbb{Z}) \times (\mathbb{Z}/(2t, 12)\mathbb{Z}).$$

Moreover, Γ_t is not a maximal discrete subgroup of $Sp_4(\mathbb{R})$. Its maximal normal extension is generated by special quasi-involutions

$$\Gamma_t^* = \langle \Gamma_t, V_d \rangle, \quad V_d^2 \in \Gamma_t, \quad d|t, \quad (d, \frac{t}{d}) = 1,$$

and $\Gamma_t^* / \Gamma_t \cong (\mathbb{Z}/2\mathbb{Z})^{\rho(t)}$, where $\rho(t)$ is the number of prime divisors of t .

3. $H^3(\Gamma_t, \mathbb{C})$

Theorem [Gr-1993]. We have $h^{3,0}(\overline{\Gamma_t \setminus \mathbb{H}_2}^{smooth}) > 0$,
if $t \neq 1, \dots, 12, 14, 15, 16, 18, 20, 24, 30, 36$.

We have canonical differential forms $F(Z)dZ$ on any smooth compactification of the moduli space $\mathcal{A}_t$, where

$$F(Z) = \text{G-Lift}(\varphi_{3,t}(\tau, z)e^{2\pi i\omega}) \in S_3(\Gamma_t).$$

For $t = 13$ we take a theta-block (see: [Gr-Skoruppa-Zagier 2026])

$$\varphi_{3,13}(\tau, z) = \frac{\vartheta(\tau, z)^5 \vartheta(\tau, 2z)^3 \vartheta(\tau, 3z)}{\eta(z)^3} \in J_{3,13}^{cusp},$$

where $\vartheta(\tau, z) = -\vartheta(\tau, -z)$ is the odd Jacobi theta-series of characteristic 2.

4. Moduli of polarised Kummer surfaces

Isomorphisms for the paramodular groups:

$$\Gamma_t / \{\pm E_4\} \cong \widetilde{SO}^+(L_t), \quad \Lambda_{2t} = U \oplus U \oplus \langle -2t \rangle, \quad \text{sign}(\Lambda_{2t}) = (2, 3).$$

$$\Gamma_t^* / \{\pm E_4\} \cong O^+(L_t) / \{\pm E_5\}.$$

Theorem [GH-1998]. $\mathcal{A}_t^* = \Gamma_t^* \backslash \mathbb{H}_2$ is the moduli space of Kummer surfaces associated to abelian surfaces with a $(1, t)$ -polarization.

To construct a canonical differential forms on $\mathcal{A}_t^*$ one needs a Γ_t -cusp form F_3 of weight 3 which is anti-symmetric with special involutions.

Problem. For what natural t the group $H^{3,0}(\mathcal{A}_t^*)$ is not trivial?

This is true ([Gr-Wang 2019]) for

$$t = p = 167, 173, 223, 227, 251, 257, 269, 271, 282, 293, \dots$$

5. Eichler transvections

Here L is an integral even lattice, i.e. a free $\mathbb{Z}$ -module with a non-degenerate bilinear form and $(u, u) = u^2 \in 2\mathbb{Z}$.

Question. *What subgroup is generated in $O(L)$ by elementary orthogonal transformations?*

Let $V = L \otimes \mathbb{Q}$ and let $e^2 = 0$ be an isotropic vector in V . The map $t'(e, a): v \mapsto v - (a, v)e \in O(e_V^\perp)$ extends uniquely to

$$t(e, a): v \mapsto v - (a, v)e + (e, v)a - \frac{1}{2}(a, a)(e, v)e \in O(V).$$

We define $E(L) < SO(L)$ to be the subgroup generated by all transvections by *unimodular* isotropic vectors:

$$E(L) := \langle \{t(e, a) \mid e, a \in L, (e, e) = (e, a) = 0, \operatorname{div}(e) = 1\} \rangle.$$

6. The spinor kernel $O'(L)$

The stable orthogonal group $\widetilde{O}(L)$ consists of elements that acts trivially on the discriminant group L^*/L .

We consider the spinor norm $\text{sn}_K : O(L \otimes K) \rightarrow K^\times / (K^\times)^2$ over a field K and define the stable orthogonal group and the spinor kernel

$$\widetilde{SO}^+(L) = \widetilde{SO}(L) \cap \ker \text{sn}_{\mathbb{R}}, \quad O'(L) = SO(L) \cap \ker \text{sn}_{\mathbb{Q}}.$$

Theorem [GHS-2009]. Let $L = U \oplus U \oplus L_0$ be an even lattice with two hyperbolic planes U with $\text{rank}_3(L) \geq 5$ and $\text{rank}_2(L) \geq 6$.

Then

$$\widetilde{SO}^+(L) = O'(L) = E(L),$$

$\widetilde{SO}^+(L)^{ab}$ is trivial and $\widetilde{O}^+(L)^{ab} \cong \mathbb{Z}/2\mathbb{Z}$.

Corollary. Any smooth model of a compactification of the moduli space $\mathcal{A}_t$ of $(1, t)$ -polarized abelian surfaces is simply-connected. The same is true for the moduli space of $2d$ -polarized K3 surfaces.

7. Kneser conditions

An even integral lattice L satisfies *the Kneser condition* if its Witt index ≥ 2 over $\mathbb{R}$, L represents -2 and that $\text{rank}_3(L) \geq 5$ and $\text{rank}_2(L) \geq 6$.

Theorem [GHS-2009]. If L satisfies the Kneser condition then

$$O'(L) = \widetilde{SO}^+(L).$$

- The abelianisation $\widetilde{O}^+(L)^{ab}$ is an abelian 2-group.
- Its order divides 2^N , where N is the number of different $\widetilde{O}^+(L)$ -orbits of (-2) -vectors in L .
- The same is true if we replace $\widetilde{O}^+(L)$ by $\widetilde{SO}^+(L)$.

To prove the theorem we used the strong approximation for indefinite orthogonal groups and some considerations with reflections.

8. Lattices of signature $(2, n)$ in geometry

$\text{sign}(L) = (2, 3)$ – the moduli spaces of polarised Abelian surfaces ($\widetilde{SO}^+(L)$) and polarised Kummer surfaces ($O^+(L)$).

$\text{sign}(L) = (2, 19)$ – the moduli spaces of polarised K3 surfaces ($\widetilde{O}^+(L)$). The last open question of A.Weil's program (1956) on K3 surfaces ([Gr-Hulek-Sankaran 2007]).

$\text{sign}(L) = (2, 20)$ – the moduli spaces of polarised hyperkähler varieties of type $\text{Hilb}^n(K3)$ ($\widetilde{O}^+(L)$) [GHS 2010, 2014].

$\text{sign}(L) = (2, 21)$ – the moduli spaces of polarised hyperkähler varieties of OG10: $\langle \widetilde{O}^+(L), \sigma \rangle$ [GHS 2011, 2014].

$\text{sign}(L) = (2, 10)$ – the moduli spaces of Enriques surfaces and of polarised Enriques surfaces [Gr-Hulek 2016].

$\text{sign}(L) = (2, 4)$ – **the moduli spaces of polarised hyperkähler varieties of type $\text{Kum}^n(A)$** . There are only rather weak results when the degree of polarisation $2d \rightarrow \infty$.

9. Hyperkähler manifolds

A **K3 surface** is the first example of irreducible holomorphic symplectic varieties (or hyperkähler manifold).

Def. A compact Kähler manifold X is called IHSV (an irreducible holomorphic symplectic variety (or **hyperkähler manifold**)) if

- 1) X is simply-connected,
- 2) $H^0(X, \Omega_X^2) = H^{2,0}(X) = \mathbb{C}\omega$ where ω is an everywhere nondegenerate holomorphic 2-form.

Beauville (1983) found two infinite series of IHSVs:

- 1) The length n Hilbert scheme $\text{Hilb}^n(S)$ for a K3 surface S .
- 2) Let A be a 2-dimensional complex torus and consider $A^{[n+1]} = \text{Hilb}^{n+1}(A)$ ($n \geq 2$) with the morphism of $p : A^{[n+1]} \rightarrow A$ given by addition. Then $X = p^{-1}(0)$ is an IHSV, called a **generalised Kummer variety**. The *deformation space* of these manifolds has dimension 5.

10. Beauville-Bogomolov lattice and polarisations

$X \sim Kum^n$: $H^2(X, \mathbb{Z}) \cong 3U \oplus \langle -2(n+1) \rangle$ ($n \geq 2$).

A **polarisation** of X is a choice of ample line bundle $\mathcal{L}$ on X

$$c_1(\mathcal{L}) = h \in H^2(X, \mathbb{Z}) \cong L_{BB}, \quad h^2 = 2d > 0.$$

For a split polarisation we get a lattice of signature $(2, 4)$

$$h_L^\perp = L_{2n+2, 2d} \cong 2U \oplus \langle -2(n+1) \rangle \oplus \langle -2d \rangle.$$

A period of X is defined as follows

$$[\varphi(\omega)] \in \Omega(L_{2n+2, 2d}) = \{[Z] \in \mathbb{P}(L_{2n+2, 2d} \otimes \mathbb{C}) : (Z, Z) = 0, (Z, \bar{Z}) > 0\}.$$

Global Torelli Theorem is true for IHSVs (Verbitski, Markmann).

In our case we have an open immersion of the moduli space

$$\mathcal{M}(Kum^n, h_{2d}^{split}) \rightarrow \Gamma_{2n+2, 2d} \setminus \Omega(L_{2n+2, 2d}),$$

$$\Gamma_{2n+2, 2d} = \langle \widetilde{SO}^+(L_{2n+2, 2d}), \sigma_{-(2n+2)} \rangle,$$

where $\sigma_{-(2n+2)}$ is the reflection with respect to the generator of $\langle -2(n+1) \rangle$ of the lattice $L_{2n+2, 2d}$ of signature $(2, 4)$.

11. Canonical differential forms on $\mathcal{M}(Kum^n, h_{2d}^{split})$

Let $F_4(\tau, z_1, z_2, \omega)$ be a modular form of weight 4 with respect to $\Gamma_{2n+2, 2d}$ with **character** $det : \Gamma_{2n+2, 2d} \rightarrow \{\pm 1\}$.

Then $F_4(Z)dZ$ is a canonical differential form on an **open part** of the modular variety. It can be continued on any its smooth compactification if and only if $F_4(Z)$ is a cusp form. Thus

$$H^{4,0}(\widetilde{\mathcal{M}}_{\Gamma_{2n+2, 2d}}) \cong S_4(\Gamma_{2n+2, 2d}, det).$$

Problem: To construct at least one such cusp form.

I can do this using my model of the **automorphic discriminant** of the moduli spaces of Enriques surfaces. This modular form belongs to the kernel of the hyperbolic differential operator for $U \oplus D_8(-1)$ and it determines a gravitational correction of the FHSV-model.

12. $H^4(\Gamma_{d,n}, \mathbb{C})$

Using the theory of theta-blocks I can construct canonical differential forms on the moduli spaces $\widetilde{\mathcal{M}}_{\Gamma_{2n+2,2d}}$ for a very wide set of pairs of dimensions $2n$ and polarisations $2d$. We give a very particular proposition having very simple formulation

Theorem [Gr 2024-26]

$$h^{4,0}(\mathcal{M}(\text{Kum}^2, h_{2d}^{\text{split}})) > 0, \quad \text{if } 2d \neq 2, 4, 6, 10, 12, 18;$$

$$h^{4,0}(\mathcal{M}(\text{Kum}^n, h_6^{\text{split}})) > 0, \quad \text{if } n \neq 2, 4, 5, 8.$$

In particular, in these cases the fourth cohomology group of $\Gamma_{2d,2n+2}$ is not trivial.